On the Brauer map of étale Galois coverings

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Introduction

The notion of the Brauer group of a field k dates back to Richard Brauer, who defined it as the set of Morita equivalence classes of central simple algebras over k. Azumaya generalized this concept to the Brauer group of a local ring and Auslander–Goldmann then defined the Brauer group of a ring. The most far-reaching generalization was presented by Grothendieck, who extended the definition to schemes. The Brauer group of a scheme X is defined as the set of Morita equivalence classes of Azumaya algebras over X, where an Azumaya algebra is an $\mathcal{O}_X$-algebra $\mathcal{A}$ that is étale locally isomorphic to a matrix algebra. Since the automorphism group of a matrix algebra $M_n(k)$ is isomorphic to the projective linear group $PGL_n(k)$ by the Skolem–Noether theorem, cocycle descriptions of Azumaya algebras allow identifying the set of isomorphism classes of Azumaya algebras of rank n^2 with the non-abelian cohomology group $H^1_{et}(X, PGL_n(\mathcal{O}_X^*))$. On the other hand, $PGL_n(k)$ can be interpreted as the automorphism group of $\mathbb{P}^{n-1}_k$. Therefore, the non-abelian cohomology group $H^1(X, PGL_n(\mathcal{O}_X^*))$ can be identified with the set of isomorphism classes of varieties $P \to X$ over X that are étale locally trivial $\mathbb{P}^{n-1}$-bundles. Varieties $P \to X$ over X with this property are called Brauer–Severi varieties. The interplay between Azumaya algebras and Brauer–Severi varieties is subject to section 1.

In section 2, we will summarize Beauville's paper [3], which calculates the Brauer groups of Enriques and K3 surfaces. In particular, the Brauer group of an Enriques surface is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. This naturally raises the question of when the induced map from the Brauer group of an Enriques surface to the Brauer group of its canonical covering is injective, or trivial, respectively. The main theorems are the following:

Theorem. *[3, Prop. 4.1] Let $\pi\colon X \to Y$ be a cyclic étale covering of a smooth projective variety Y over an algebraically closed field k. Denote by σ a generator of the deck transformation group G of π, and denote by* Nm *the norm homomorphism* $\mathrm{Nm}\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$. *Then there is an isomorphism*

$$\ker(\pi^{Br}\colon Br(Y) \to Br(X)) \cong \ker(\mathrm{Nm})/\mathrm{im}(\mathrm{id} - \sigma^*).$$

Corollary. *[3, Cor. 4.3] Let $\pi\colon X \to Y$ be the universal covering of an Enriques surface Y over $\mathbb{C}$. Then the following conditions are equivalent:*

(i) *The map π^{Br} on Brauer groups is trivial,*

(ii) *there exists a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ such that $\pi_* c_1(\mathcal{L}) = 0$ in $H^2(Y, \mathbb{Z})$ and $c_1(\mathcal{L}) \notin \mathrm{im}(\mathrm{id} - \sigma^*) \subset H^2(X, \mathbb{Z})$.*

In section 3, we investigate the Brauer map of an étale Galois covering $\pi\colon X \to Y$ with Galois group G, where X and Y are smooth projective varieties over an algebraically closed field k. The differentials on the second page of the Hochschild–Serre spectral sequence

$$E_2^{p,q} := H^p(G, H^q_{et}(X, \mathcal{O}_X^*)) \Rightarrow H^{p+q}_{et}(Y, \mathcal{O}_Y^*)$$

induce a seven-term exact sequence

$$0 \to \ker(\pi^*) \to \mathrm{Pic}(Y) \xrightarrow{\pi^*} H^0(G, \mathrm{Pic}(X)) \to H^2(G, k^*) \xrightarrow{I} \ker(\pi^{Br}) \xrightarrow{\Theta} H^1(G, \mathrm{Pic}(X)) \\ \to H^3(G, k^*).$$

We propose a geometric description of Θ and I. The map I constructs Brauer–Severi varieties from normalized 2-cocycles $\lambda\colon G \times G \to k^*$. The map Θ can be viewed as a generalization of a construction by Martínez, [18, Lem. 10]:

Theorem. *Let $f\colon G \to \mathrm{Pic}(X)$ be a crossed homomorphism such that its class is sent to zero by the map $H^1(G, \mathrm{Pic}(X)) \to H^3(G, k^*)$. Then one can find a G-action on $\mathbb{P}(\bigoplus_{g\in G} f(g))$, which commutes with the G-action on X such that the quotient $P := \mathbb{P}(\bigoplus_{g\in G} f(g))/G$ defines a Brauer–Severi variety over Y whose Brauer class is mapped to $[f]$ by Θ.*

Furthermore, we will examine the situation where G is cyclic more closely. In this case, $H^2(G, k^*) = 0$, which implies that Θ is injective.

Theorem. *Suppose that G is cyclic of order d and suppose that $\mathrm{Pic}(X)[d] = 0$, where $[d]$ denotes the kernel of multiplication by d. Then the map Θ is an isomorphism.*

Theorem. *Suppose that G is cyclic of prime order d. Then the sequence*

$$0 \to \ker(\pi^{Br}) \to H^1(G, \mathrm{Pic}(X)) \stackrel{\mathrm{Nm}}{\to} \ker(\pi^*) \cap \mathrm{im}([d]) \to 0$$

is exact.

In section 4, we study Enriques surfaces Y over $\mathbb{C}$ with the property that the canonical covering $\pi\colon X \to Y$ induces the trivial map on the Brauer groups. Denote by $\sigma\colon X \to X$ the Enriques involution and let $\mathcal{L} \in \mathrm{Pic}(X)$ be a non-trivial line bundle such that $\sigma^*\mathcal{L} \cong \mathcal{L}^\vee$. We embed the Brauer–Severi variety constructed by Martínez as a conic bundle $P \hookrightarrow \mathbb{P}(\mathcal{O}_Y \oplus \pi_*\mathcal{L})$, and show that every smooth conic bundle $C \in |\mathcal{O}_{\mathbb{P}(\mathcal{O}_Y\oplus\pi_*\mathcal{L})}(2)|$ is isomorphic to P, and that the Brauer class of P is non-trivial if and only if $c_1^2(\mathcal{L}) \equiv 2 \mod 4$. In addition, we show that replacing $\mathcal{O}_Y$ by a line bundle $\mathcal{M}$ and $\pi_*\mathcal{L}$ by a stable vector bundle $\mathcal{F}$ of rank 2 does not yield a larger class of Brauer–Severi varieties $C \hookrightarrow \mathbb{P}(\mathcal{M} \oplus \mathcal{F})$:

Proposition. *Let $\mathcal{M} \in \mathrm{Pic}(Y)$ be a line bundle and let $\mathcal{F}$ be a stable vector bundle over Y of rank 2 for a fixed polarization $\mathcal{O}_Y(1) \in \mathrm{Pic}(Y)$. Suppose there is a conic bundle $C \hookrightarrow \mathbb{P}(\mathcal{M} \oplus \mathcal{F})$, such that all fibers of $C \to Y$ are smooth. Then there is a line bundle $\mathcal{N}$ such that*

$$\mathcal{N}^\vee \otimes (\mathcal{M} \oplus \mathcal{F}) \cong \mathcal{O}_Y \oplus \pi_*\mathcal{L},$$

where $\mathcal{L} \in \mathrm{Pic}(X)$ is an anti-invariant line bundle. Moreover, the corresponding isomorphism $\mathbb{P}(\mathcal{M} \oplus \mathcal{F}) \stackrel{\sim}{\to} \mathbb{P}(\mathcal{O}_Y \oplus \pi_*\mathcal{L})$ sends C to the previously constructed conic bundle.

Furthermore, we apply Beauville's theorem to the moduli spaces $M_Y((2, 0, -2n), \omega_Y)$ of semistable vector bundles on Y:

Theorem. *The map π^{Br} is injective if and only if $M_Y((2, 0, -2n), \omega_Y)$ is smooth for all $n \geq 1$.*

In [6] Ferrari, Tiribassi, and Vodrup apply Beauville's theorem to bielliptic surfaces over $\mathbb{C}$ and their canonical coverings. In section 5 we will give a short introduction to bielliptic surfaces and prove the main results of [6] concerning the Brauer map of the canonical covering of bielliptic surfaces of type 1, 3, and 5. Furthermore, explicit examples will be presented.

Conventions

By a variety, we mean an integral, separated scheme X of finite type over an algebraically closed field k. If X is smooth of dimension d, we denote by $\omega_X := \Omega_X^d$ the canonical line bundle. Let $\mathcal{E}$ be a vector bundle over a variety X. The projective bundle $\mathbb{P}(\mathcal{E}) \to X$ over X is defined as

$$\mathbb{P}(\mathcal{E}) := \mathrm{Proj}_X(\mathrm{Sym}^\bullet(\mathcal{E}^\vee)) \to X,$$

where $\mathcal{E}^\vee$ denotes the dual bundle of $\mathcal{E}$.

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1 The Brauer group of a variety

1.1 Azumaya algebras

Definition 1.1. [9, 1.2] Let X be a scheme. A sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras is called an *Azumaya algebra* if there exists an étale cover $\{\varphi_i\colon U_i \to X\}$ such that $\varphi_i^*\mathcal{A} \cong \mathcal{M}_n(\mathcal{O}_{U_i})$ as $\mathcal{O}_{U_i}$-algebras, where $\mathcal{M}_n(\mathcal{O}_X)$ denotes the sheaf of $n \times n$ matrices. Two Azumaya algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* if there exist locally free sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\mathcal{A}_1 \otimes \mathcal{E}nd(\mathcal{E}_1) \cong \mathcal{A}_2 \otimes \mathcal{E}nd(\mathcal{E}_2)$ as $\mathcal{O}_X$-algebras.

Definition 1.2. [9, 1.2] The *algebraic Brauer group* $Br^{alg}(X)$ of a scheme X is defined as the set of equivalence classes of Azumaya algebras over X. The group operation is given by the tensor product and the neutral element is the equivalence class of $\mathcal{O}_X$.

Note that the canonical isomorphism

$$\mathcal{A} \otimes \mathcal{A}^{opp} \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}),\ \sum a_i \otimes b_i \mapsto [x \mapsto \sum a_i x b_i],$$

[9, 1.2], implies that the inverse of the Brauer class of an Azumaya algebra in $Br^{alg}(X)$ is the Brauer class of its opposite algebra (which is an Azumaya algebra as well), and thus the algebraic Brauer group is a group. Since the tensor product is commutative up to isomorphism, the Brauer group is abelian. Applying the Skolem–Noether isomorphism

$$PGL_n \xrightarrow{\sim} Aut(M_n),\ \overline{A} \mapsto [M \mapsto AMA^{-1}],$$

[9, 1.1], to a cocycle description of an Azumaya algebra implies that the set of isomorphism classes of Azumaya algebras of rank n^2 is in bijection to $H^1_{et}(X, PGL_n(\mathcal{O}_X))$. The short exact sequences $0 \to \mathcal{O}_X^* \to GL_n(\mathcal{O}_X) \to PGL_n(\mathcal{O}_X) \to 0$ induces maps

$$H^1_{et}(X, GL_n(\mathcal{O}_X)) \to H^1_{et}(X, PGL_n(\mathcal{O}_X)) \xrightarrow{\delta} H^2_{et}(X, \mathcal{O}_X^*)$$

for all n, [9, 1.3]. Since the fiber of δ over the zero element consists of trivial Azumaya algebras, taking the colimit over n yields an injection $Br^{alg}(X) \hookrightarrow H^2_{et}(X, \mathcal{O}_X^*)$. The commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_n & \longrightarrow & SL_n(\mathcal{O}_X^*) & \longrightarrow & PGL_n(\mathcal{O}_X^*) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & GL_n(\mathcal{O}_X^*) & \longrightarrow & PGL_n(\mathcal{O}_X^*) & \longrightarrow & 0 \end{array}$$

induces the commutative square

$$\begin{array}{ccc} H^1_{et}(X, PGL_n(\mathcal{O}_X^*)) & \longrightarrow & H^2_{et}(X, \mu_n) \\ \downarrow & & \downarrow \\ H^1_{et}(X, PGL_n(\mathcal{O}_X)) & \longrightarrow & H^2_{et}(X, \mathcal{O}_X^*), \end{array}$$

where μ_n denotes the étale sheaf that assigns to each étale morphism $U \to X$ the multiplicative subgroup of the n-th roots of unity in $\Gamma(U, \mathcal{O}_U)$. Since $H^2_{et}(X, \mu_n)$ is n-torsion, the class of an Azumaya algebra of rank n^2 is n-torsion as well, and hence the algebraic Brauer group is a torsion group, [9, 1.3].

Definition 1.3. [10, Rem. 2.7] Let X be a scheme. The *cohomological Brauer group* $Br^{coh}(X)$ is defined as the étale cohomology group $H^2_{et}(X, \mathcal{O}_X^*)$.

The highly interesting question of when the algebraic and cohomological Brauer groups coincide has been answered in only very few cases. The case of smooth projective curves over an algebraically closed field follows from Tsen's theorem, [19, III. Ex. 2.22d]. The case of a smooth surface was settled by Grothendieck:

Theorem 1.4. *[10, Thm. 2.1] Let X be a noetherian scheme, and let $\alpha \in H^2_{et}(X, \mathcal{O}_X^*)$. Then there is a locally closed subscheme $Y \subset X$ of codimension at least two, such that $\alpha|_{X \setminus Y}$ comes from $Br(X \setminus Y)$. Moreover, if X is regular, then one can demand that the codimension of Y is at least three.*

This shows the equality of the algebraic and the cohomological Brauer groups for any curves and smooth surfaces. Hoobler proved the equality in the case of abelian varieties, [12, Thm. 3.3]. In [7], Gabber showed equality in the case of the union of two affine schemes glued over an affine scheme.

In the case where X is a smooth projective variety over $\mathbb{C}$, one can use the analytic topology instead of the étale topology to define Azumaya algebras.

Lemma 1.5. *Let X be a smooth projective variety over $\mathbb{C}$ and let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. The following conditions are equivalent:*

(i) *There is an open cover $X^{an} = \bigcup U_i$ in the analytic topology such that $\mathcal{A}|_{U_i} \cong \mathcal{M}_n(\mathcal{O}_{U_i})$ as an $\mathcal{O}_{U_i}$-algebra.*

(ii) *$\mathcal{A}$ is locally free in the Zariski topology and the canonical map $\mathcal{A} \otimes \mathcal{A}^{opp} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A})$ is an isomorphism.*

(iii) *$\mathcal{A}$ is an Azumaya algebra.*

Proof. By [2, I. Thm. 19.1], every $\mathcal{O}_X$-module that is locally free in the analytic topology is also locally free in the Zariski topology. The map $\mathcal{A} \otimes \mathcal{A}^{opp} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A})$ is an isomorphism if and only if it is an isomorphism at all closed points. If $\mathcal{A}|_{U_i} \cong \mathcal{M}_n(\mathcal{O}_{U_i})$ as an $\mathcal{O}_{U_i}$-algebra, the map $\mathcal{A} \otimes \mathcal{A}^{opp} \otimes \kappa(x) \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}) \otimes \kappa(x)$ can be identified with

$$M_n(\kappa(x)) \otimes M_n(\kappa(x))^{opp} \to End(M_n(\kappa(x))),$$

which is an isomorphism. This proves $(i) \Rightarrow (ii)$.

To prove the implication $(ii) \Rightarrow (iii)$ we follow [19, IV. Prop. 2.1]. Let $x\colon \mathrm{Spec}(K) \to X$ be a geometric point and define $A := x^*\mathcal{A}$. By assumption, the map

$$A \otimes A^{opp} \to End_K(A)$$

is an isomorphism. Consider a two-sided ideal $0 \neq I \subset A$. The image of $I \otimes I$ is a two-sided ideal in $End_K(A)$, which thus has to be equal to $End_K(A)$. Therefore, $I = A$, which proves that A is simple. Let $C \subset A$ be the center of A. Note that C is also the center of A^{opp}, and the image of $C \otimes C$ is contained in the center of $End_K(A)$, which equals K. Hence, $C = K$, and A is a central simple algebra. The Brauer group of an algebraically closed field is zero, and hence A is isomorphic to a matrix algebra. Thus, an étale cover $U \to X$ with x contained in its image, along which $\mathcal{A}$ pulls back to a matrix algebra, exists.

The implication $(iii) \Rightarrow (i)$ trivial. □

Proposition 1.6. *Suppose that X is a smooth projective variety over $\mathbb{C}$. The canonical morphism*

$$\psi\colon H^2_{et}(X, \mathcal{O}_X^*) \to H^2_{an}(X, \mathcal{O}_X^*)$$

is injective with image equal to the torsion subgroup of $H^2_{an}(X, \mathcal{O}_X^*)$.

Proof. For each n the Kummer sequence $0 \to \mathbb{Z}/n\mathbb{Z} \to \mathcal{O}_X^* \xrightarrow{\cdot n} \mathcal{O}_X^* \to 0$ induces this commutative diagram with exact rows:

$$\begin{array}{ccccccc} \mathrm{Pic(X)} & \longrightarrow & H^2_{et}(X,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^2_{et}(X,\mathcal{O}_X^*) & \xrightarrow{\cdot n} & H^2_{et}(X,\mathcal{O}_X^*) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ \mathrm{Pic(X)} & \longrightarrow & H^2_{an}(X,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^2_{an}(X,\mathcal{O}_X^*) & \xrightarrow{\cdot n} & H^2_{an}(X,\mathcal{O}_X^*) \end{array}$$

The map φ is an isomorphism by the comparison theorem [19, III. Thm. 3.12], and $H^2_{et}(X,\mathcal{O}_X^*)$ is a torsion group by [10, Prop. 1.4]. By choosing n appropriately, one can show the injectivity and surjectivity of ψ by diagram chases. □

1.2 Brauer–Severi varieties

A more geometric way to interpret elements of $H^1_{et}(X, PGL_n(\mathcal{O}_X))$ is the notion of a Brauer–Severi variety.

Definition 1.7. [9, 8] Let X be a variety over k. A *Brauer–Severi variety* over X is a variety $p\colon P \to X$ such that there is an étale cover $\{\varphi_i\colon U_i \to X\}$ trivializing π:

$$\begin{array}{ccccc} \mathbb{P}^{n-1}\times U_i & \xrightarrow{\sim} & P\times_X U_i & \longrightarrow & P \\ & \searrow^{pr} & \downarrow & & \downarrow{\scriptstyle p} \\ & & U_i & \longrightarrow & X. \end{array}$$

A Brauer–Severi variety $p\colon P \to X$ is called *trivial* if it is the projectivization of a vector bundle.

Since the automorphism group of $\mathbb{P}^{n-1}_k$ is given by $PGL_n(k)$, one obtains a bijection between the isomorphism classes of Brauer–Severi varieties over X of fiber dimension $n-1$ and the cohomology group $H^1_{et}(X, PGL_n(\mathcal{O}_X))$, using cocycle descriptions. Furthermore, note that a Brauer–Severi variety is trivial if and only if its cocycle defines an element in the fiber of

$$\delta\colon \mathrm{colim}_n H^1_{et}(X, PGL_n(\mathcal{O}_X)) \to H^2_{et}(X,\mathcal{O}_X^*)$$

over the zero element.

Lemma 1.8. *[8, 8.13] Let X be a variety over k and let $\mathcal{A}$ be an Azumaya algebra of rank n^2 over X. Then the closed subvariety $P_{\mathcal{A}} \subset Gr(n,\mathcal{A})$ of rank n subbundles that are left ideals (with the reduced induced scheme structure) is a Brauer–Severi variety that defines the same class as $\mathcal{A}$ in $H^1_{et}(X, PGL_n(\mathcal{O}_X))$.*

Proof. Let $\{\varphi_i\colon U_i \to X\}$ be an étale cover of X that trivializes $\mathcal{A}$. The pullback $P_{\mathcal{A}} \times_X U_i$ is isomorphic to the subvariety of rank n left ideals of $\mathcal{M}_n(\mathcal{O}_{U_i})$ via the isomorphism
$\psi_i\colon \varphi_i^*\mathcal{A} \xrightarrow{\sim} \mathcal{M}_n(\mathcal{O}_{U_i})$. Suppose that $\mathcal{I} \subset \mathcal{M}_n(\mathcal{O}_{U_i})$ is a left ideal of rank n. Denote by e_{ij} the $n\times n$ matrix with a one in the ij-coordinate and zeros everywhere else. Then $\mathcal{I}$ can be written as

$$\mathcal{I} = \bigoplus_{i=0}^{n} e_{ii}\mathcal{I}.$$

Suppose that $e_{ii}\mathcal{I}$ has rank of at least two for some i. Choose an arbitrary j and consider the left ideal $e_{ji}\mathcal{I}$. The injection $e_{ji}\mathcal{I} \hookrightarrow \mathcal{I}$ respects the splitting, and thus induces $e_{ji}\mathcal{I} = e_{jj}e_{ji}\mathcal{I} \hookrightarrow e_{jj}\mathcal{I}$, since $e_{jj}e_{ji} = e_{ji}$. Note that multiplying $e_{ii}\mathcal{I}$ by e_{ji} interchanges the i-th and j-th row, and hence preserves the rank. Therefore, the rank of $e_{jj}\mathcal{I}$ is at least two for each j, which contradicts the rank of $\mathcal{I}$. Consequently, $e_{ii}\mathcal{I}$ has rank at most one for all i, and thus has rank equal to one for all i. Choose a generator of $e_{11}\mathcal{I}$ and denote by v its first row. The matrix that has v in its i-th row and zeros everywhere else generates $e_{ii}\mathcal{I}$. Therefore, $v \in \Gamma(U_i, \mathcal{O}_{U_i}) \setminus \{0\}$ determines $\mathcal{I}$. Conversely, any non-zero local section $v \in \Gamma(U_i, \mathcal{O}_{U_i})$ generates a left ideal of rank n via this correspondence. We therefore conclude that the pullback $P_{\mathcal{A}} \times_X U_i$ is isomorphic to $\mathbb{P}^{n-1}_{U_i}$, and hence $P_{\mathcal{A}}$ defines a Brauer–Severi variety. Recall that $\mathcal{A}$ was given by the isomorphisms $\psi_i \colon \varphi_i^* \mathcal{A} \xrightarrow{\sim} \mathcal{M}_n(\mathcal{O}_{U_i})$. Over $U_i \times_X U_j$ the cocycle of $P_{\mathcal{A}} \times_X U_i \times_X U_j$ is thus given by $\psi_j|_{U_{ij}} \circ \psi_i^{-1}|_{U_{ij}}$, and hence $P_{\mathcal{A}}$ defines the same cohomology class as $\mathcal{A}$ in $H^1_{et}(X, PGL_n(\mathcal{O}_X))$. □

In [17, Thm. 74], Kollár gives an inverse of this construction: Let $p \colon P \to X$ be a Brauer–Severi variety of rank $n-1$, and let $\{\varphi_i \colon U_i \to X\}$ be an étale cover of X that trivializes P via isomorphisms $\psi_i \colon P \times_X U_i \to \mathbb{P}_{U_i}$. Over each U_i one can form the relative Euler sequence

$$0 \to \Omega_{\mathbb{P}^{n-1}_{U_i}/U_i} \to \mathcal{O}^{\oplus n}_{\mathbb{P}^{n-1}_{U_i}}(-1) \to \mathcal{O}_{\mathbb{P}^{n-1}_{U_i}} \to 0.$$

These sequences can be glued to the global relative Euler sequence. After dualizing and pushing forward on X, this sequence results in

$$0 \to \mathcal{O}_X \to p_*\mathcal{A} \to \mathcal{T}_{P/X} \to 0.$$

The sequence is right exact, since $R^1 p_* \mathcal{O}_P = 0$. Note that $p_*\mathcal{A}$ is an Azumaya algebra of rank n^2. Indeed, étale locally the sequence looks like

$$0 \to \mathcal{O}_{U_i} \to \mathcal{E}nd(\mathcal{O}^{\oplus n}_{U_i}) \to \mathcal{T}_{\mathbb{P}_{U_i}/U_i} \to 0,$$

where the first map sends $1 \in \Gamma(U_i, \mathcal{O}_{U_i})$ to the identity. Moreover, the maps $\psi_j \circ \psi_i^{-1}$ are given by multiplication by a matrix due to the Skolem–Noether theorem, and hence the induced maps on $\mathcal{E}nd(\mathcal{O}^{\oplus n}_{U_{ij}})$ are given by conjugation with the inverse of this matrix. Therefore, the algebra structure of $\mathcal{E}nd(\mathcal{O}^{\oplus n}_{U_i})$ is preserved, which shows that $p_*\mathcal{A}$ is an Azumaya algebra of rank n^2. The canonical map $\mathcal{A} \to p^* p_* \mathcal{A}$ allows to embed P into $Gr(n, p_*\mathcal{A})$ as the subvariety of rank n right ideals. After transposing, one obtains $P \hookrightarrow Gr(n, p_*\mathcal{A}^{opp})$, which equals the map from the previous lemma.

The following well-known statement allows us to replace the étale setting with the analytic setting.

Lemma 1.9. *Let X be a smooth projective variety over $\mathbb{C}$. The morphism $p \colon P \to X$ is a Brauer–Severi variety if and only if $p^{an} \colon P^{an} \to X^{an}$ is a $\mathbb{P}^{n-1}$-fiber bundle.*

Proof. If $p \colon P \to X$ is locally trivial in the étale topology, then it is locally trivial in the analytic topology by standard arguments.

Suppose that $P^{an} \to X^{an}$ is locally trivial. Then one can form the relative Euler sequence as above to obtain the $\mathcal{O}_X$-algebra $p_*\mathcal{A}$ and the embedding $P \hookrightarrow Gr(n, p_*\mathcal{A}^{opp})$. Since $p_*\mathcal{A}$ is an Azumaya algebra by lemma 1.5, P is a Brauer–Severi variety by lemma 1.8. □

1.3 Twisted Sheaves

Another possibility to interpret elements in $H^1_{et}(X, PGL_n(\mathcal{O}_X))$ is as twisted sheaves.

Definition 1.10. [13, Ch. 16, 5] Let X be a variety and let $\alpha \in Br(X)$ be represented by the cocycle (α_{ijk}) on an étale cover $\{\varphi_i \colon U_i \to X\}$. An (α_{ijk})-twisted sheaf is a collection of coherent sheaves $\mathcal{F}_i$ on U_i and transition homomorphisms $\varphi_{ij} \colon \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}}$ such that $\varphi_{ii} = \mathrm{id}$, $\varphi_{ij} = \varphi_{ji}^{-1}$ and $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

Definition 1.11. [13, Ch. 16, 5] Let $\mathcal{F}$ and $\mathcal{G}$ be (α_{ijk})-twisted sheaves over a variety X with transition homomorphisms φ_{ij} and ψ_{ij}, respectively. A morphism $f\colon \mathcal{F} \to \mathcal{G}$ of twisted coherent sheaves is a collection of homomorphisms $f_i\colon \mathcal{F}_i \to \mathcal{G}_i$ such that $\psi_{ij} \circ f_i|_{U_{ij}} = f_i|_{U_{ij}} \circ \varphi_{ij}$.

Remark. [13, Ch. 16, 5] Note that the choice of a different representative of α yields an equivalent category and thus it makes sense to define the category $Coh(X, \alpha)$ of twisted sheaves on X. Moreover, this category is abelian.

Let $p\colon P \to X$ be a Brauer–Severi variety that trivializes over $\{\varphi_i\colon U_i \to X\}$:

$$\begin{array}{ccc} P \times_X U_i & \xrightarrow{f_i} & \mathbb{P}^{n-1} \times U_i \\ & \searrow \quad \swarrow & \\ & U_i & \end{array}$$

with cocycle $f_{ij} \in PGL_n(U_{ij})$ and let α be the Brauer class of the associated cocycle $\delta(f_{ij})$. Consider $\mathcal{O}_i(1) := f_i^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Choosing preimages $g_{ij} \in GL_n(\mathcal{O}_{U_{ij}})$ of f_{ij} induces isomorphisms

$$g_{ij}\colon \pi_* \mathcal{O}_i(1)|_{U_{ij}} \xrightarrow{\sim} \pi_* \mathcal{O}_j|_{U_{ij}}$$

that satisfy $g_{ij} \circ g_{jk} \circ g_{ki} = \alpha_{ijk} \cdot \mathrm{id}$. Thus, we obtain the α-twisted sheaf $\pi_* \mathcal{O}(1)$.

On the other hand, let $\mathcal{F}$ be an α-twisted sheaf on X with transition maps $\varphi_{ij}\colon \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}}$. One can glue the projective bundles $\mathbb{P}(\mathcal{F}_i) \to U_i$ via the transition maps φ_{ij}, since the scalar α_{ijk} vanishes after the projectivization. This results in a Brauer–Sever variety $\mathbb{P}(\mathcal{F}) \to X$ with Brauer class α.

Note that these constructions are inverse to each other.

2 Summary of Beauville's Paper

This thesis is based on Beauville's paper [3], which will be summarized in this section.

Theorem 2.1. *[3, 2] Let X be a smooth projective surface over $\mathbb{C}$, let $H^2(X,\mathbb{Z})_{tf}$ be the quotient of $H^2(X,\mathbb{Z})$ by its torsion subgroup, and let $T_X \subset H^2(X,\mathbb{Z})_{tf}$ be the orthogonal complement of the image of* $\mathrm{Pic}(X)$ *in $H^2(X,\mathbb{Z})_{tf}$ with respect to the cup-product. Then there is a short exact sequence*

$$0 \to \mathrm{Hom}(T_X, \mathbb{Q}/\mathbb{Z}) \to Br(X) \to H^3(X,\mathbb{Z})_{torsion} \to 0. \tag{1}$$

Proof. Applying cohomology to the Kummer exact sequence $0 \to \mathbb{Z}/n\mathbb{Z} \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0$ and to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ yields exact sequences

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Z}/n\mathbb{Z} \to H^2(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{p} Br(X)[n] \to 0 \tag{2}$$

and

$$0 \to H^2(X,\mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z} \to H^2(X,\mathbb{Z}/n\mathbb{Z}) \to H^3(X,\mathbb{Z})[n] \to 0, \tag{3}$$

where $[n]$ denotes the kernel of the multiplication by n. The commutativity of the diagram

$$\begin{array}{ccccc} H^1(X,\mathcal{O}_X^*) & \xrightarrow{\cdot n} & H^1(X,\mathcal{O}_X^*) & \longrightarrow & H^2(X,\mathbb{Z}/n\mathbb{Z}) \\ \downarrow c_1 & & \downarrow c_1 & & \downarrow \mathrm{id} \\ H^2(X,\mathbb{Z}) & \xrightarrow{\cdot n} & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathbb{Z}/n\mathbb{Z}) \end{array}$$

implies that the map $\mathrm{Pic}(X) \otimes \mathbb{Z}/n\mathbb{Z} \to H^2(X,\mathbb{Z}/n\mathbb{Z})$ in (2) factors through the first map in (3). Combining these exact sequences thus results in the long exact sequence

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Z}/n\mathbb{Z} \to H^2(X,\mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z} \to Br(X)[n] \to H^3(X,\mathbb{Z})[n] \to 0. \tag{4}$$

By definition of T_X, there exists an exact sequence

$$\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to T_X^\vee \to 0.$$

Tensoring with $\mathbb{Z}/n\mathbb{Z}$ yields

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Z}/n\mathbb{Z} \to H^2(X,\mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z} \to T_X^\vee \otimes \mathbb{Z}/n\mathbb{Z} \to 0, \tag{5}$$

which is left exact since the first map agrees with the first map in (2). By combining this sequence with (4), one obtains

$$0 \to \mathrm{Hom}(T_X, \mathbb{Z}/n\mathbb{Z}) \to Br(X)[n] \to H^3(X,\mathbb{Z})[n] \to 0,$$

and taking the direct limit over n yields the desired sequence. □

Definition 2.2. [2, VI. 1] A *K3 surface* is a smooth projective surface X over $\mathbb{C}$ such that

(i) $\omega_X \cong \mathcal{O}_X$

(ii) $H^1(X, \mathcal{O}_X) = 0$.

An *Enriques surface* is a smooth projective surface Y over $\mathbb{C}$ such that

(i) $\omega_Y^2 \cong \mathcal{O}_Y$, $\omega_Y \not\cong \mathcal{O}_Y$

(ii) $H^1(Y, \mathcal{O}_Y) = 0$.

Corollary 2.3. *[3, 2]*

(i) *If X is a K3 surface, then the map* $\mathrm{Hom}(T_X, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} Br(X)$ *is an isomorphism.*

(ii) *If Y is an Enriques surface, then the map* $Br(Y) \to H^3(Y, \mathbb{Z})_{torsion} \cong \mathbb{Z}/2\mathbb{Z}$ *is an isomorphism.*

Proof. The first statement follows from the fact that $H^3(X, \mathbb{Z}) = 0$. Indeed, Poincaré duality implies $H^3(X, \mathbb{Z}) \cong H_1(X, \mathbb{Z}) \cong \pi(X)^{ab} = 0$, since all K3 surfaces are simply connected, [2, VIII. Cor. 8.6].

Similarly, Poincaré duality implies $H^3(Y, \mathbb{Z}) \cong \pi(Y)^{ab} \cong \mathbb{Z}/2\mathbb{Z}$, [2, VIII. Lem. 15.1]. On the other hand, the exponential sequence shows $T_Y = 0$. □

Theorem 2.4. *[3, Prop. 4.1] Let $\pi\colon X \to Y$ be a cyclic étale covering of a smooth projective variety Y over an algebraically closed field k. Denote by σ a generator of the deck transformation group G of π, and denote by $\mathrm{Nm}\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ the norm homomorphism. Then there is an isomorphism*

$$\ker(\pi^{Br}\colon Br(Y) \to Br(X)) \cong \ker(\mathrm{Nm})/\mathrm{im}(\mathrm{id} - \sigma^*).$$

Proof. The Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q_{et}(X, \mathcal{O}_X^*)) \Rightarrow H^{p+q}_{et}(Y, \mathcal{O}_Y^*)$$

identifies $\ker(\pi^{Br})$ with $\ker(d_2^{1,1}\colon E_2^{1,1} \to E_2^{3,0})$, since $H^2(G, \mathbb{C}^*) = 0$. The periodicity of group cohomology implies $E_2^{3,0} \cong E_2^{1,0} \cong \mathrm{Hom}(G, \mathbb{C}^*)$, [4, VI. 9.2]. Therefore, we obtain

$$d_2^{1,1}\colon H^1(G, \mathrm{Pic}(X)) \to \mathrm{Hom}(G, \mathbb{C}^*).$$

On the other hand, consider the map

$$f\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X),\ \mathcal{L} \mapsto \bigotimes_{g \in G} g^*\mathcal{L},$$

and note that $H^1(G, \mathrm{Pic}(X))$ is isomorphic to $\ker(f)/\mathrm{im}(\mathrm{id} - \sigma^*)$, [4, III. 1. Ex. 2]. Since $\pi^*\mathrm{Nm}(\mathcal{L}) \cong f(\mathcal{L})$ and since $\mathrm{Nm} \circ (\mathrm{id} - \sigma^*) = 0$, we obtain

$$\overline{\mathrm{Nm}}\colon H^1(G, \mathrm{Pic}(X)) \to \ker(\pi^*).$$

Recall that $\ker(\pi^*)$ is isomorphic to $\mathrm{Hom}(G, \mathbb{C}^*)$: a character $\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$ corresponds to the subsheaf $\mathcal{L}_\chi \subset \pi_*\mathcal{O}_X$ where G acts via χ. It thus suffices to show that the two maps $\overline{\mathrm{Nm}}$ and $d_2^{1,1}$ agree.

Note that there is a natural isomorphism $H^p(G, -) \cong \mathrm{Ext}^p_{\mathbb{Z}[G]}(\mathbb{Z}, -)$. We can describe the map $d^{1,1}$ using [24, Prop. 1.1]: The Yoneda pairing

$$\smile\colon \mathrm{Ext}^1_{\mathbb{Z}[G]}(G, \mathrm{Pic}(X)) \times \mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathrm{Pic}(X), \mathbb{C}^*) \to \mathrm{Ext}^3_{\mathbb{Z}[G]}(G, \mathbb{C}^*)$$

yields $d_2^{1,1}(x) = x \smile \partial^{0,1}(\mathrm{id})$ for all $x \in \mathrm{Ext}^1_{\mathbb{Z}[G]}(G, \mathrm{Pic}(X))$, where $\partial^{0,1}$ is the map

$$\partial^{0,1}\colon \mathrm{Hom}_{\mathbb{Z}[G]}(\mathrm{Pic}(X), \mathrm{Pic}(X)) \to \mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathrm{Pic}(X), \mathbb{C}^*)$$

that comes from the spectral sequence $\mathrm{Ext}^p_{\mathbb{Z}[G]}(\mathrm{Pic}(X), H^q_{et}(X, \mathcal{O}_X^*)) \Rightarrow \mathrm{Ext}^{p+q}(\mathrm{Pic}(X), \mathcal{O}_Y^*)$. Observe that $\partial^{0,1}(\mathrm{id})$ is given by the exact sequence

$$0 \to \mathbb{C}^* \to R_X^* \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0,$$

where R_X denotes the field of rational functions on X. Henceforth, $d_2^{1,1}$ can be identified with the composite

$$H^1(G,\mathrm{Pic}(X)) \to H^2(G,R_X^*/\mathbb{C}^*) \to H^3(G,\mathbb{C}^*) \cong \mathrm{Hom}(G,\mathbb{C}^*),$$

where the first map is the boundary map associated to $0 \to R_X^*/\mathbb{C}^* \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0$ and the second map is the boundary map associated to $0 \to \mathbb{C}^* \to R_X^* \to R_X^*/\mathbb{C}^* \to 0$. Suppose that $x \in H^1(G,\mathrm{Pic}(X))$ is represented by $x = [\mathcal{L}]$ for a line bundle $\mathcal{L} \in \ker(\mathrm{Nm})$. Choose a divisor $D \in \mathrm{Div}(X)$ such that $\mathcal{O}_X(D) \cong \mathcal{L}$. Note that $\sum_g g^*D$ is the divisor of a rational function $\psi \in R_X^*$. The class $\overline{\psi} \in R_X^*/\mathbb{C}^*$ is well-defined and invariant under G. Therefore, the first map of the composite above sends x to $[\psi] \in H^2(G,R_X^*/\mathbb{C}^*)$. Since $\mathrm{div}(\psi)$ is invariant under G there is a character χ such that $g^*\psi = \chi(g)\psi$ for all $g \in G$. Consequently, the map $d_2^{1,1}$ sends x to χ.

To see that $\overline{\mathrm{Nm}}$ sends x to χ it suffices to show $\mathcal{O}_Y(\pi_*D) \cong \mathcal{L}$. Since $\pi^*\pi_*D = \mathrm{div}(\psi)$ we obtain a global isomorphism $u\colon \pi^*\mathcal{O}_Y(\pi_*D) \xrightarrow{\sim} \mathcal{O}_X$ by multiplication with ψ. Let $U \subset X$ be a G-invariant open subset and let $\varphi \in R_X$ be a generator of $\mathcal{O}_X(D)|_U$. Then $\mathrm{Nm}(\varphi)$ is a generator of $\mathcal{O}_Y(\pi_*D)|_{\pi(U)}$ and $\pi^*\mathrm{Nm}(\varphi)$ is a generator of $\pi^*\mathcal{O}_Y(\pi_*D)|_U$. Define $h := \psi \cdot \pi^*\mathrm{Nm}(\varphi)$, and note

$$g^*h = g^*\psi \cdot g^*\pi^*\mathrm{Nm}(\varphi) = \chi(g)\psi \cdot \pi^*\mathrm{Nm}(\varphi) = \chi(g)h$$

for all $g \in G$. Therefore, the adjoint of u is the map $\mathcal{O}_Y(\pi_*D) \to \pi_*\mathcal{O}_X$, which has image equal to $\mathcal{L}_\chi$. □

Corollary 2.5. *[3, Cor. 4.3] Let $\pi\colon X \to Y$ be the universal covering of an Enriques surface Y. Then the following conditions are equivalent:*

(i) *The map π^{Br} on Brauer groups is trivial,*

(ii) *there exists a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ such that $\pi_*c_1(\mathcal{L}) = 0$ in $H^2(Y,\mathbb{Z})$ and $c_1(\mathcal{L}) \notin \mathrm{im}(\mathrm{id} - \sigma^*) \subset H^2(X,\mathbb{Z})$.*

Proof. Using theorem 2.4, the proof reduces to showing that the map

$$H^1(c_1)\colon H^1(\mathbb{Z}/2\mathbb{Z},\mathrm{Pic}(X)) \to H^1(\mathbb{Z}/2\mathbb{Z},H^2(X,\mathbb{Z}))$$

is injective. The exponential sequence yields the short exact sequence

$$0 \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to Q \to 0,$$

where $Q \subset H^2(X,\mathcal{O}_X)$. Note that $H^2(Y,\mathcal{O}_Y) = 0$, and hence the only invariant element in $H^2(X,\mathcal{O}_X)$ is zero. This implies $H^0(\mathbb{Z}/2\mathbb{Z},Q) = 0$, and the associated long exact sequence yields then the desired injectivity. □

Beauville then examines the cup products on $H^2(Y,\mathbb{Z})$ and $H^2(Y,\mathbb{Z}/2\mathbb{Z})$ to find an equivalent condition for the existence of a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ such that $\pi_*c_1(\mathcal{L}) = 0$ and $c_1(\mathcal{L}) \notin \mathrm{im}(\mathrm{id} - \sigma^*)$.

Denote by E the lattice $E_8(-1) \oplus H$, where H is the rank 2 hyperbolic lattice. Then one can find isomorphisms $H^2(Y,\mathbb{Z})_{tf} \cong E$ and $H^2(X,\mathbb{Z}) \cong E \oplus E \oplus H$ that identify the map $\pi^*\colon H^2(Y,\mathbb{Z})_{tf} \to H^2(X,\mathbb{Z})$ with the diagonal embedding $\delta\colon E \hookrightarrow E \oplus E$, and identify the involution $\sigma^*\colon H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ with

$$\rho\colon E \oplus E \oplus H \to E \oplus E \oplus H,\ (x,y,z) \mapsto (y,x,-z),$$

[2, VIII Lem. 19.1].

Definition 2.6. [3, 5] Let M be an even lattice, and define $M_2 := M/2M$. Then there is an associated quadratic form $q\colon M_2 \to \mathbb{Z}/2\mathbb{Z}$ given by $q(\overline{m}) := (\overline{\tfrac{m^2}{2}})$.

Lemma 2.7. *[3, Lem. 5.4] For all $x \in H^2(Y, \mathbb{Z}/2\mathbb{Z})$ we have $q(\pi^* x) = x^2$.*

Proof. Note that the universal coefficient theorem implies $H^2(X, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \cong H^2(X, \mathbb{Z}/2\mathbb{Z})$, [11, Thm. 3.2]. We use this to calculate $\mathcal{P}(\pi^* x)$, where $\mathcal{P}\colon H^2(X, \mathbb{Z}/2\mathbb{Z}) \to H^4(X, \mathbb{Z}/4\mathbb{Z})$ is the Pontryagin square, [20, Ch. 2 Ex. 1]:

$$\mathcal{P}(\pi^* x) = 2\mathcal{P}(x) = 2x^2$$

by functoriality, and

$$\mathcal{P}(\pi^* x) = 2q(\pi^* x),$$

since $\pi^* x \in H^2(X, \mathbb{Z}/2\mathbb{Z})$ comes from $H^2(X, \mathbb{Z})$. □

Proposition 2.8. *[3, Prop. 3.5]*

(i) *The kernel of $\pi^*\colon H^2(Y, \mathbb{Z}/2\mathbb{Z}) \to H^2(X, \mathbb{Z}/2\mathbb{Z})$ consists of $\{0, k_Y\}$, where k_Y denotes the class of the canonical divisor K_Y on Y.*

(ii) *The map $\pi_*\colon H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is surjective.*

Proof. Consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}/2\mathbb{Z}, H^q(X, \mathbb{Z}/2\mathbb{Z})) \Rightarrow H^{p+q}(Y, \mathbb{Z}/2\mathbb{Z}).$$

Note that $E_\infty^{0,2} = E_2^{0,2} = H^2(X, \mathbb{Z}/2\mathbb{Z})$, $E_\infty^{1,1} = E_2^{1,1} = 0$ and $E_\infty^{2,0} = E_2^{2,0} = \mathbb{Z}/2\mathbb{Z}$. Therefore, the kernel of $\pi^*\colon H^2(Y, \mathbb{Z}/2\mathbb{Z}) \to H^2(X, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and since it contains k_Y the statement follows.

The map π_* is given by the composite

$$H^2(X, \mathbb{Z}) \xrightarrow{[X]\frown} H_2(X, \mathbb{Z}) \xrightarrow{\pi_*} H_2(Y, \mathbb{Z}) \xrightarrow{([Y]\frown)^{-1}} H^2(Y, \mathbb{Z}),$$

where $[X]$ and $[Y]$ are the fundamental classes. For $x \in H^2(Y, \mathbb{Z})$ the first two maps send $\pi^* x$ to $\pi_*([X] \frown \pi^* x) = \pi_*[X] \frown x = 2[Y] \frown x$, since π has degree 2, and hence $\pi_* \pi^* x$ equals $2x$. The cokernel of π_* is thus a $\mathbb{Z}/2\mathbb{Z}$-module, and it suffices to show that the induced map $t_{\pi_*}\colon \mathrm{Hom}(H^2(Y, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(H^2(X, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$ is injective. Consider the diagram

$$\begin{array}{ccccc} H^2(Y, \mathbb{Z}/2\mathbb{Z}) & \twoheadrightarrow & \mathrm{Hom}(H_2(Y, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Hom}(H^2(Y, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \circ\pi_*} & & \downarrow{\scriptstyle t_{\pi_*}} \\ H^2(X, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Hom}(H^2(X, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}). \end{array}$$

The left square is obtained from the universal coefficient theorem [11, Thm. 3.2]. The right square is deduced from Poincaré duality, and hence the right vertical map equals t_{π_*}. Consider an element in the kernel of t_{π_*}, and let $x \in H^2(Y, \mathbb{Z}/2\mathbb{Z})$ be a preimage. Since the outer diagram commutes and since the lower row is an isomorphism, $x \in \ker(\pi^*) = \{0, k_Y\}$. The upper row can be identified with the map

$$H^2(Y, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(H^2(Y, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}),\ y \mapsto [z \mapsto y \smile \overline{z}].$$

In both cases, the form $z \mapsto x \smile \overline{z}$ is the zero form, and hence the kernel of t_{π_*} is trivial. □

Define $\varepsilon \in H_2 := H/2H$ as the class of $e + f$, where (e, f) is a hyperbolic basis of H. Note that this is the unique element in H_2 on which q evaluates to 1.

Proposition 2.9. *[3, Prop. 5.3] The image of $\pi^*\colon H^2(Y, \mathbb{Z}/2\mathbb{Z}) \to H^2(X, \mathbb{Z}/2\mathbb{Z})$ equals $\delta(E_2) \oplus (\mathbb{Z}/2\mathbb{Z})\varepsilon$.*

Proof. Since the image of π^* is σ^*-invariant, it is contained in $\delta(E_2) \oplus H_2$. By proposition 2.8, the image is 11-dimensional, and is thus a hyperplane in $\delta(E_2) \oplus H_2$ that contains $\delta(E_2)$. It hence suffices to find the unique nonzero element $\pi^* x \in \operatorname{im}(\pi^*) \cap H_2$. By the previous lemma, we find $1 = x^2 = q(\pi^* x)$, thus $\pi^* x = \varepsilon$. □

Corollary 2.10. *[3, Cor. 5.5] The map $\pi_* \colon H_2 \to H^2(Y, \mathbb{Z}/2\mathbb{Z})$ takes image in $\{0, k_Y\}$ and its kernel equals $\{0, \varepsilon\}$.*

Proof. First, note that σ^* acts by multiplication with (-1) on H. Therefore, for $x \in H$ we have $\pi_* x = \pi_* \sigma^* x = -\pi_* x$, and thus the map $\pi_* \colon H \to H^2(Y, \mathbb{Z})$ takes image in $\{0, K_Y\}$. Consequently, the image of $\pi_* \colon H_2 \to H^2(Y, \mathbb{Z}/2\mathbb{Z})$ lies in $\{0, k_Y\}$.

Since $\pi_* \pi^* x = 2x = 0$ for all $x \in H^2(Y, \mathbb{Z}/2\mathbb{Z})$, we have $\pi_* \varepsilon = 0$. By proposition 2.8, there is an element $y \in H^2(X, \mathbb{Z})$ such that $\pi_* y = K_Y$. Note that $y + \sigma^* y$ is σ^* invariant, thus corresponding to an element of the form $(z, z, 0) \in E \oplus E \oplus H$. Since $\pi^* \colon H^2(Y, \mathbb{Z})_{tf} \to H^2(X, \mathbb{Z})$ corresponds to the diagonal embedding, we have $y + \sigma^* y = \pi^* z$. Therefore, $2z = \pi_* \pi^* z = \pi_*(y + \sigma^* y) = K_Y + K_Y = 0$, and hence $y + \sigma^* y = (z, z, 0) = 0$. Consequently, $\overline{y}$ defines an element in $H^1(\mathbb{Z}/2\mathbb{Z}, H^2(X, \mathbb{Z}))$ with $\pi_* \overline{y} \neq 0$. To conclude, note that $H^1(\mathbb{Z}/2\mathbb{Z}, H^2(X, \mathbb{Z})) \cong H_2$ by the same argument as in theorem 2.5. □

Corollary 2.11. *[3, Cor. 5.6] Let $x \in H^2(X, \mathbb{Z})$. Then the following conditions are equivalent:*

(i) $\pi_* x = 0$ *and* $x \notin \operatorname{im}(\mathrm{id} - \sigma^*)$.

(ii) $\sigma^* x = -x$ *and* $x^2 \equiv 2 \mod 4$.

Proof. The element $x + \sigma^* x$ is σ^* invariant and thus of the form $(a, a, 0)$. The condition $\pi_* x = 0$ implies $0 = \pi_*(x + \sigma^* x) = 2a$, and therefore $x + \sigma^* x = 0$, equivalently $\sigma^* x = -x$. If $x = y - \sigma^* y$, writing $y = (b, c, d)$ yields $x = y - \sigma^* y = (b - c, c - b, 2d)$. Therefore, condition (i) is equivalent to $x = (\alpha, -\alpha, \beta)$ such that $0 \neq \overline{\beta} \in H_2$ and $\pi_* \overline{\beta} = 0$. Since $\pi_* \overline{\beta} = 0$ if and only if $\overline{\beta} = \varepsilon$ by the previous corollary, we conclude that (i) is equivalent to $x = (\alpha, -\alpha, \beta)$ such that $\overline{\beta} = \varepsilon$.

On the other hand, (ii) is equivalent to $x = (\alpha, -\alpha, \beta)$ such that

$$2 \equiv x^2 = 2\alpha^2 + \beta^2 \equiv 2q(\overline{\beta}) \mod 4,$$

which is equivalent to $\overline{\beta} = \varepsilon$. □

This allows to reformulate corollary 2.5:

Theorem 2.12. *[3, Cor. 5.7] The following conditions are equivalent:*

(i) *The map $\pi^{Br} \colon Br(Y) \to Br(X)$ is trivial.*

(ii) *There is a line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ such that $\sigma^* \mathcal{L} \cong \mathcal{L}^\vee$ and $c_1(\mathcal{L})^2 \equiv 2 \mod 4$.*

3 On the seven-term sequence of an étale Galois covering

Let $\pi\colon X \to Y$ be an étale Galois covering with Galois group G, where X and Y are smooth projective varieties over an algebraically closed field k. The differentials on the second page of the Hochschild–Serre spectral sequence

$$E_2^{p,q} := H^p(G, H^q_{et}(X, \mathcal{O}_X^*)) \Rightarrow H^{p+q}_{et}(Y, \mathcal{O}_Y^*)$$

induce a seven-term exact sequence

$$0 \to E_2^{1,0} \to \mathrm{Pic}(Y) \to E_2^{0,1} \to E_2^{2,0} \xrightarrow{I} \ker(\pi^{Br}) \xrightarrow{\Theta} E_2^{1,1} \to E_2^{3,0}.$$

We will describe the maps Θ and I in terms of Brauer–Severi varieties, which allows constructing Brauer–Severi varieties from crossed homomorphisms $f\colon G \to \mathrm{Pic}(X)$ whenever their class is mapped to zero via $E_2^{1,1} \to E_2^{3,0}$ as well as from cohomology classes $[\lambda] \in H^2(G, k^*)$.

Remark. Strictly speaking, there is a G^{opp}-action and not a G-action on $H^q_{et}(X, \mathcal{O}_X^*)$, cohomology is contravariant in the first variable. Therefore, the Hochschild–Serre spectral sequence should read

$$E_2^{p,q} := H^p(G^{opp}, H^q_{et}(X, \mathcal{O}_X^*)) \Rightarrow H^{p+q}_{et}(Y, \mathcal{O}_Y^*).$$

To facilitate the readability, however, we drop this from the notation.

The construction of the maps Θ and I uses the following two statements:

Lemma 3.1. *Let $\pi\colon X \to Y$ be an étale Galois covering with Galois group G, and let $p\colon P \to X$ be a Brauer–Severi variety over X. Then P fits into a fiber product diagram*

$$\begin{array}{ccc} P & \xrightarrow{\pi'} & Q \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\pi} & Y \end{array}$$

if and only if the G-action on X can be lifted to a G-action on P.

Proof. On the one hand, if P is a pullback then it comes with deck transformations, which form the G-action. On the other hand, a G-action on P as above is fixed point free, since π is étale. Therefore, the quotient Q defines a Brauer–Severi variety Q over Y, which pulls back to P. □

Lemma 3.2 (Universal property of Proj)**.** *[5, Prop. 9.2] Let $f\colon Y \to X$ be a morphism of smooth projective varieties, and let $\mathcal{E}$ be a vector bundle on X. Then there is a natural bijection of the set of morphisms $g\colon Y \to \mathbb{P}(\mathcal{E})$ making the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\quad g \quad} & \mathbb{P}(\mathcal{E}) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & X & \end{array}$$

commute with the set of line subbundles $\mathcal{L} \subset f^*\mathcal{E}$ via the map

$$g \mapsto g^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \subset f^*\mathcal{E}.$$

Proof. Let $\mathcal{L} \subset f^*\mathcal{E}$ be a line subbundle, and choose an open cover of Y that trivializes $\mathcal{L}$ and $f^*\mathcal{E}$. The universal property of the projective space allows us to define local maps to $\mathbb{P}(\mathcal{E})$, which agree on intersections since they are unique. Thus, we can glue these maps to a global map $g\colon Y \to \mathbb{P}(\mathcal{E})$. □

To construct I, consider a cohomology class $[\lambda] \in H^2(G, k^*)$, represented by the normalized 2-cocycle $\lambda\colon G \times G \to k^*$, [4, IV. 3]. Since the G-action on k^* is trivial, λ satisfies the relation

$$\lambda(f,g)\lambda(f*g,h) = \lambda(g,h)\lambda(f,g*h)$$

for all f, g, $h \in G$, where $*$ denotes the opposite multiplication. Define

$$\mathcal{E}_\lambda := \bigoplus_{h\in G} h^*\mathcal{O}_X,$$

and for each $g \in G$ define the map $A_\lambda(g)\colon g^*\mathcal{E}_\lambda \to \mathcal{E}_\lambda$ as the direct sum

$$A_\lambda(g) := \bigoplus_{h\in G}(g^*h^*\mathcal{O}_X \xrightarrow{\lambda(g,h)} (g*h)^*\mathcal{O}_X).$$

From now on, we consider the maps $A_\lambda(g)$ as part of the data of $\mathcal{E}_\lambda$.

Lemma 3.3. *For g_1, $g_2 \in G$ we have $A_\lambda(g_1) \circ (g_1)^*A_\lambda(g_2) = \lambda(g_1,g_2)A_\lambda(g_1 * g_2)$.*

Proof. Let $h \in G$. On $h^*\mathcal{O}_X$ the map $A_\lambda(g_1) \circ (g_1)^*A_\lambda(g_2)$ is given by

$$(g_1)^*(g_2)^*h^*\mathcal{O}_X \xrightarrow{\lambda(g_2,h)} (g_1)^*(g_2*h)^*\mathcal{O}_X \xrightarrow{\lambda(g_1,g_2*h)} (g_1*g_2*h)^*\mathcal{O}_X.$$

On the other hand, the map $A_\lambda(g_1 * g_2)$ is given by

$$(g_1*g_2)^*h^*\mathcal{O}_X \xrightarrow{\lambda(g_1*g_2,h)} (g_1*g_2*h)^*\mathcal{O}_X.$$

Since $\lambda(g_1,g_2)\lambda(g_1*g_2,h) = \lambda(g_2,h)\lambda(g_1,g_2*h)$, the assertion follows. □

By the universal property of Proj, these maps allow us to lift the G-action on X to a G-action on $\mathbb{P}(\mathcal{E}_\lambda)$: the map $\tau_\lambda(g)\colon \mathbb{P}(\mathcal{E}_\lambda) \to \mathbb{P}(\mathcal{E}_\lambda)$ is given by

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda)}(-1) \subset p^*\mathcal{E}_\lambda \xrightarrow{p^*A_\lambda^{-1}(g)} p^*g^*\mathcal{E}_\lambda.$$

Definition 3.4. We define $P_\lambda \to Y$ as the quotient of $\mathbb{P}(\mathcal{E}_\lambda) \to X$ by this G-action, and we define $I([\lambda])$ as the Brauer class of P_λ.

Lemma 3.5. (i) *The variety $P_\lambda \to Y$ is a Brauer–Severi variety whose Brauer class $[P_\lambda]$ lies in the kernel of π^{Br}.*

(ii) *The isomorphism class of $P_\lambda \to Y$ only depends on the cohomology class $[\lambda] \in H^2(G,k^*)$.*

(iii) *The assignment $[\lambda] \mapsto I([\lambda])$ is a well-defined group homomorphism $I\colon H^2(G,k^*) \to \ker(\pi^{Br})$.*

Proof. The first assertion follows from lemma 3.1.

Let $\lambda\colon G \times G \to k^*$ be a normalized 2-cocycle, and let $\mu\colon G \to k^*$ be an arbitrary map. The coboundary $\mathrm{d}\mu$ is then given by $\mathrm{d}\mu(g,h) = \mu(g)\mu^{-1}(g*h)\mu(h)$. To show the first assertion it suffices to construct an isomorphism $P_{\lambda\cdot\mathrm{d}\mu} \xrightarrow{\sim} P_\lambda$. Define the map

$$M := \bigoplus_{h\in G}(h^*\mathcal{O}_X \xrightarrow{\mu(h)} h^*\mathcal{O}_X)\colon \mathcal{E}_{\lambda\cdot\mathrm{d}\mu} \to \mathcal{E}_\lambda,$$

and note that $A_{\lambda\cdot \mathrm{d}\mu}(g) = \mu(g)\cdot M^{-1}\circ A_\lambda(g)\circ g^*M$ for all $g\in G$. Consider the following diagram, where the horizontal morphism f is induced by $\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\lambda\cdot \mathrm{d}\mu})}(-1)\subset p^*\mathcal{E}_{\lambda\cdot\mathrm{d}\mu}\stackrel{p^*M}{\to} p^*\mathcal{E}_\lambda$:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_{\lambda\cdot\mathrm{d}\mu}) & \xrightarrow{f} & \mathbb{P}(\mathcal{E}_\lambda) \\ \downarrow{\scriptstyle \tau_{\lambda\cdot\mathrm{d}\mu}(g)} & & \downarrow{\scriptstyle \tau_\lambda(g)} \\ \mathbb{P}(\mathcal{E}_{\lambda\cdot\mathrm{d}\mu}) & \xrightarrow{f} & \mathbb{P}(\mathcal{E}_\lambda). \end{array}$$

Note that the composition through the upper right corner is given by

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\lambda\cdot\mathrm{d}\mu})}(-1)\subset p^*\mathcal{E}_{\lambda\cdot\mathrm{d}\mu}\xrightarrow{p^*M} p^*\mathcal{E}_\lambda \xrightarrow{p^*A_\lambda^{-1}(g)} p^*g^*\mathcal{E}_\lambda,$$

and that the composition through the lower left corner is given by

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\lambda\cdot\mathrm{d}\mu})}(-1)\subset p^*\mathcal{E}_{\lambda\cdot\mathrm{d}\mu}\xrightarrow{p^*A^{-1}_{\lambda\cdot\mathrm{d}\mu}(g)} p^*g^*\mathcal{E}_{\lambda\cdot\mathrm{d}\mu}\xrightarrow{p^*g^*M} p^*g^*\mathcal{E}_\lambda.$$

Thus, since $A_{\lambda\cdot\mathrm{d}\mu}(g) = \mu(g)\cdot M^{-1}\circ A_\lambda(g)\circ g^*M$, the diagram commutes. Therefore, f is G-equivariant, and consequently descends to an isomorphism $\overline{f}\colon P_{\lambda\cdot\mathrm{d}\mu}\xrightarrow{\sim} P_\lambda$.

To show the second assertion, let λ and $\mu\colon G\times G\to k^*$ be two cocycles, and denote by $\mathbf{1}\colon G\times G\to k^*$ the constant map with value 1. Note that the identity $\mathrm{id}\colon \mathcal{E}_\lambda\otimes\mathcal{E}_\mu\to\mathcal{E}_{\lambda\cdot\mu}\otimes\mathcal{E}_{\mathbf{1}}$ induces a G-equivariant morphism $\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{E}_\mu)\xrightarrow{\sim}\mathbb{P}(\mathcal{E}_{\lambda\cdot\mu}\otimes\mathcal{E}_{\mathbf{1}})$. Since $\mathcal{E}_{\mathbf{1}}$ descends to $\mathcal{O}_Y^{\oplus|G|}$, we obtain

$$I([\lambda][\mu]) = I([\lambda\cdot\mu]) = [\mathbb{P}(\mathcal{E}_{\lambda\cdot\mu})/G] = [\mathbb{P}(\mathcal{E}_{\lambda\cdot\mu}\otimes\mathcal{E}_{\mathbf{1}})/G] = [\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{E}_\mu)/G] = I([\lambda])I([\mu]).$$

□

Lemma 3.6. *The kernel of I equals the image of $H^0(G,\mathrm{Pic}(X))\to H^2(G,k^*)$.*

Proof. Let $\mathcal{L}\in\mathrm{Pic}(X)$ be G-invariant. Choose isomorphisms $\varphi_g\colon g^*\mathcal{L}\xrightarrow{\sim}\mathcal{L}$. There are scalars $\lambda(g,h)\in k^*$ such that

$$\varphi_g\circ g^*\varphi_h = \lambda(g,h)\varphi_{g*h}$$

for all g, $h\in G$. The map λ is a cocycle, and the line bundle $\mathcal{L}$ is sent to the cohomology class of λ by the map $H^0(G,\mathrm{Pic}(X))\to H^2(G,k^*)$. It thus suffices to show that the Brauer class of $P_\lambda\to Y$ is trivial. To see this, we construct a G-action on $\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)$ together with a G-equivariant isomorphism $f\colon\mathbb{P}(\mathcal{E}_\lambda)\xrightarrow{\sim}\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)$, and show that the Brauer class of $\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)/G$ is trivial. Define the maps

$$B(g) := A_\lambda(g)\otimes(\varphi_g^\vee)^{-1}\colon g^*\mathcal{E}_\lambda\otimes g^*\mathcal{L}^\vee\to\mathcal{E}_\lambda\otimes\mathcal{L}^\vee,$$

and note that $B(g)\circ g^*B(h) = B(g*h)$ for all g, $h\in G$. By $\tilde\tau(g)\colon\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)\to\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)$ we denote the maps induced by $\mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)}(-1)\subset p^*(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)\xrightarrow{p^*B^{-1}(g)} p^*g^*(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)$. Consider the following diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_\lambda) & \xrightarrow{f} & \mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee) \\ \downarrow{\scriptstyle \tau_\lambda(g)} & & \downarrow{\scriptstyle \tilde\tau(g)} \\ \mathbb{P}(\mathcal{E}_\lambda) & \xrightarrow{f} & \mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee), \end{array}$$

where f is the canonical map induced by $\mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda}(-1)\otimes p^*\mathcal{L}^\vee\subset p^*(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)$. Note that the composite through the upper right corner is given by

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda}(-1)\otimes p^*\mathcal{L}^\vee\subset p^*(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)\xrightarrow{p^*B^{-1}(g)} p^*g^*(\mathcal{E}_\lambda\otimes\mathcal{L}^\vee)$$

and that the composition through the lower left corner is given by

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda}(-1) \otimes p^*\mathcal{L}^\vee \subset p^*\mathcal{E}_\lambda \otimes p^*\mathcal{L}^\vee \overset{p^*A_\lambda^{-1}(g)\otimes p^*\varphi_g^\vee}{\longrightarrow} p^*g^*\mathcal{E}_\lambda \otimes p^*g^*\mathcal{L}^\vee = p^*g^*(\mathcal{E}_\lambda \otimes \mathcal{L}^\vee).$$

Since both compositions are equal, the diagram commutes. Thus, f descends to an isomorphism $\overline{f}\colon P_\lambda \xrightarrow{\sim} \mathbb{P}(\mathcal{E}_\lambda \otimes \mathcal{L}^\vee)/G$. Since $B(g) \circ g^*B(h) = B(g * h)$ for all g, $h \in G$, the vector bundle $\mathcal{E}_\lambda \otimes \mathcal{L}^\vee$ descends to a vector bundle $\mathcal{F}$ over Y, and hence $P_\lambda \cong \mathbb{P}(\mathcal{F})$, which shows that $I([\lambda])$ is trivial.

Now, suppose there is a cocycle λ such that the Brauer class of P_λ is trivial. This is the case if and only if there is a vector bundle $\mathcal{F}$ over Y such that P_λ is isomorphic to $\mathbb{P}(\mathcal{F})$. Denote by $\varphi_g\colon g^*\pi^*\mathcal{F} \xrightarrow{\sim} \pi^*\mathcal{F}$ the G-action, which induces the descent to $\mathcal{F}$, and denote by $\sigma(g)$ the induced G-action on $\mathbb{P}(\pi^*\mathcal{F})$. We can find a line bundle $\mathcal{L}$ and an isomorphism $\psi\colon \mathcal{E}_\lambda \otimes \mathcal{L} \xrightarrow{\sim} \pi^*\mathcal{F}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_\lambda \otimes \mathcal{L}) & \xrightarrow{a} & \mathbb{P}(\mathcal{E}_\lambda) \\ {\scriptstyle c}\downarrow & \swarrow{\scriptstyle b} & \\ \mathbb{P}(\pi^*\mathcal{F}), & & \end{array}$$

where a is induced by $\mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L})}(-1) \otimes p^*\mathcal{L}^\vee \subset p^*\mathcal{E}_\lambda$, b is the pullback of the isomorphism $P_\lambda \cong \mathbb{P}(\mathcal{F})$, and c is induced by ψ. For all $g \in G$ we define the morphism $\tilde{\tau}(g)\colon \mathbb{P}(\mathcal{E}_\lambda \otimes \mathcal{L}) \to \mathbb{P}(\mathcal{E}_\lambda \otimes \mathcal{L})$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{E}_\lambda \otimes \mathcal{L}) & \xrightarrow{a} & \mathbb{P}(\mathcal{E}_\lambda) & \xrightarrow{b} & \mathbb{P}(\pi^*\mathcal{F}) \\ \downarrow{\scriptstyle \tilde{\tau}(g)} & & \downarrow{\scriptstyle \tau_\lambda(g)} & & \downarrow{\scriptstyle \sigma(g)} \\ \mathbb{P}(\mathcal{E}_\lambda \otimes \mathcal{L}) & \xrightarrow{a} & \mathbb{P}(\mathcal{E}_\lambda) & \xrightarrow{b} & \mathbb{P}(\pi^*\mathcal{F}). \end{array}$$

The composite $\tilde{\tau}(g) = a^{-1} \circ \tau_\lambda(g) \circ a$ is given by

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L})}(-1) \otimes p^*\mathcal{L}^\vee \subset p^*(\mathcal{E}_\lambda \otimes \mathcal{L}) \otimes p^*\mathcal{L}^\vee = p^*\mathcal{E}_\lambda \overset{p^*A_\lambda^{-1}(g)}{\longrightarrow} p^*g^*\mathcal{E}_\lambda = p^*g^*(\mathcal{E}_\lambda \otimes \mathcal{L}) \otimes p^*g^*\mathcal{L}^\vee.$$

On the other hand, the composite $\tilde{\tau}(g) = c^{-1} \circ \sigma(g) \circ c$ is given by

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L})}(-1) \subset p^*(\mathcal{E}_\lambda \otimes \mathcal{L}) \overset{p^*\psi}{\longrightarrow} p^*\pi^*\mathcal{F} \overset{p^*\sigma^{-1}(g)}{\longrightarrow} p^*g^*\pi^*\mathcal{F} \overset{p^*g^*\psi^{-1}}{\longrightarrow} p^*g^*(\mathcal{E}_\lambda \otimes \mathcal{L}).$$

Tensoring both inclusions with $\mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L})}(1)$ and the first additionally with $p^*\mathcal{L}$, we obtain

$$\begin{aligned} \mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L})} &\subset p^*g^*(\mathcal{E}_\lambda \otimes \mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L})}(1) \otimes p^*g^*\mathcal{L}^\vee \otimes p^*\mathcal{L},\ 1 \mapsto s \\ \mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L})} &\subset p^*g^*(\mathcal{E}_\lambda \otimes \mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_\lambda\otimes\mathcal{L})}(1),\ 1 \mapsto t. \end{aligned}$$

Since both maps agree, there has to be a nowhere vanishing global section $t_0 \in H^0(X, g^*\mathcal{L}^\vee \otimes \mathcal{L})$ such that $s = t \otimes t_0$. In other words, there has to be a map $B(g)\colon g^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ such that $\psi^{-1} \circ \varphi_g \circ \psi = A_\lambda(g) \otimes B(g)$. Note that these maps satisfy

$$B(g) \circ g^*B(h) = \lambda^{-1}(g, h)B(g * h)$$

for all g, $h \in G$. Thus, the map $H^0(G, \mathrm{Pic}(X)) \to H^2(G, k^*)$ sends $\mathcal{L}^\vee$ to $[\lambda]$. □

To construct the map Θ, consider a Brauer class $\alpha \in \ker(\pi^{Br})$. By [12, Prop. 3.1], one can choose a Brauer–Severi variety P_α such that its Brauer class equals α, since $\pi^*\alpha = 0$. Choose a vector bundle $\mathcal{E}_\alpha$ on X such that the diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_\alpha) & \longrightarrow & P_\alpha \\ \downarrow{\scriptstyle p} & & \downarrow \\ X & \xrightarrow{\pi} & Y \end{array}$$

is a fiber product diagram, and denote by $\tau_g\colon \mathbb{P}(\mathcal{E}_\alpha) \to \mathbb{P}(\mathcal{E}_\alpha)$ the G-action on $\mathbb{P}(\mathcal{E}_\alpha)$. Note that the choice of $\mathcal{E}_\alpha$ induces the splitting $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_\alpha)) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$. Consequently, τ_g^* on $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_\alpha))$ is of the form

$$\tau_g^* = \begin{pmatrix} g^* & \varphi_g \\ 0 & \psi \end{pmatrix},$$

where $\varphi_g\colon \mathbb{Z} \to \mathrm{Pic}(X)$ and $\psi\colon \mathbb{Z} \to \mathbb{Z}$ are homomorphisms. Note that τ_g induces an isomorphism between two fibers, which are both projective spaces. On any fiber it sends $\mathcal{O}_{\mathbb{P}^n}(1)$ to itself, and hence $\psi = \mathrm{id}$.

Lemma 3.7. *The map $f_\alpha\colon G \to \mathrm{Pic}(X)$, $g \mapsto \varphi_g(1)$ is a crossed homomorphism.*

Proof. Denote by $*$ the opposite multiplication on G. For g, $h \in G$, one calculates

$$\begin{aligned} f_\alpha(g * h) &= \varphi_{hg}(1) = pr(\tau_{hg}^*\mathcal{O}_{\mathbb{P}(\mathcal{E}_\alpha)}(1)) = pr(\tau_g^*\tau_h^*(\mathcal{O}_{\mathbb{P}(\mathcal{E}_\alpha)}(1)) = pr(\tau_g^*(\mathcal{O}_{\mathbb{P}(\mathcal{E}_\alpha)}(1) \otimes p^*\varphi_h(1))) \\ &= \varphi_g(1) \otimes g^*\varphi_h(1) = f_\alpha(g) + g^* f_\alpha(h), \end{aligned}$$

where $pr\colon \mathrm{Pic}(\mathbb{P}(\mathcal{E}_\alpha)) \to \mathrm{Pic}(X)$ is the projection. □

Definition 3.8. We define $\Theta(\alpha) := [f_\alpha] \in H^1(G, \mathrm{Pic}(X))$.

Lemma 3.9. *The map Θ is well-defined.*

Proof. Two choices were made in the construction: the choice of P_α and the choice of $\mathcal{E}_\alpha$.

Let Q be another Brauer–Severi variety that represents α. Choose twisted sheaves $\mathcal{F}_{1,2}$ such that $P_\alpha \cong \mathbb{P}(\mathcal{F}_1)$ and $Q \cong \mathbb{P}(\mathcal{F}_2)$. Then there are vector bundle $\mathcal{G}_1$, $\mathcal{G}_2$ on Y such that $\mathcal{F}_1 \otimes \mathcal{G}_1 \cong \mathcal{F}_2 \otimes \mathcal{G}_2$, and thus

$$\pi^*\mathcal{F}_1 \otimes \pi^*\mathcal{G}_1 \cong \pi^*\mathcal{F}_2 \otimes \pi^*\mathcal{G}_2.$$

Therefore, it suffices to show that replacing $\mathcal{E}_\alpha$ by $\mathcal{E}_\alpha \otimes \pi^*\mathcal{G}$ for any vector bundle $\mathcal{G}$ on Y does not change $\Theta(\alpha)$. This is obvious.

Let $\pi^*P_\alpha \cong \mathbb{P}(\mathcal{E})$ be another choice of $\mathcal{E}_\alpha$. Then there is a line bundle $\mathcal{L}$ such that $\mathcal{E}_\alpha \cong \mathcal{E} \otimes \mathcal{L}$. This isomorphism affects φ_g by replacing $\varphi_g(1)$ by $\varphi_g(1) \otimes \mathcal{L}^\vee \otimes g^*\mathcal{L}$ for all g. However, both crossed homomorphisms define the same class in $H^1(G, \mathrm{Pic}(X))$, since they differ by the principal cross homomorphism $g \mapsto \mathcal{L}^\vee \otimes g^*\mathcal{L}$. □

Lemma 3.10. *The map Θ is a group homomorphism.*

Proof. Let α, $\beta \in \ker(\pi^{Br})$ and choose representatives P_α, P_β. Suppose that $\pi^*P_\alpha \cong \mathbb{P}(\mathcal{E}_\alpha)$, $\pi^*P_\beta \cong \mathbb{P}(\mathcal{E}_\beta)$, and denote by

$$\begin{pmatrix} g^* & \varphi_g \\ 0 & \mathrm{id} \end{pmatrix}, \quad \begin{pmatrix} g^* & \psi_g \\ 0 & \mathrm{id} \end{pmatrix}$$

the actions on $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_\alpha))$, resp. $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_\beta))$. Then $\pi^*P_{\alpha\cdot\beta} \cong \mathbb{P}(\mathcal{E}_\alpha \otimes \mathcal{E}_\beta)$, which comes with automorphisms τ_g such that

$$\tau_g^* = \begin{pmatrix} g^* & \varphi_g \otimes \psi_g \\ 0 & \mathrm{id} \end{pmatrix},$$

which proves the claim. □

Lemma 3.11. *The kernel of Θ equals the image of I.*

Proof. It is clear from the constructions of I and Θ that the composite $\Theta \circ I$ is trivial. It hence suffices to show that every Brauer class in the kernel of Θ lies in the image of I.

Suppose there is a Brauer class $\alpha \in \ker(\pi^{Br})$ such that $\Theta(\alpha) = 0$. Choose a representative P_α such that $\pi^* P_\alpha \cong \mathbb{P}(\mathcal{E}_\alpha)$. Let τ_g^* be of the form

$$\tau_g^* = \begin{pmatrix} g^* & \varphi_g \\ 0 & \mathrm{id} \end{pmatrix}.$$

Since the class of $f_\alpha \colon g \mapsto \varphi_g(1)$ is trivial, there is a line bundle $\mathcal{L}$ such that f_α equals the cross homomorphism $g \mapsto \mathcal{L}^\vee \otimes g^*\mathcal{L}$. Replacing $\mathcal{E}_\alpha$ with $\mathcal{E}_\alpha \otimes \mathcal{L}^\vee$ changes the τ_g^*'s to

$$(\tau')_g^* = \begin{pmatrix} g^* & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Therefore, the τ_g''s come from isomorphisms $B(g) \colon g^*(\mathcal{E}_\alpha \otimes \mathcal{L}^\vee) \xrightarrow{\sim} \mathcal{E}_\alpha \otimes \mathcal{L}^\vee$, which are uniquely determined up to scalar. Hence, there is a function $\lambda \colon G \times G \to k^*$ such that

$$B(g) \circ g^* B(h) = \lambda(g,h) B(g * h)$$

for all g, $h \in G$. Applying f^* yields the equation

$$\lambda(f,g)\lambda(f * g, h) = \lambda(g,h)\lambda(f, g * h)$$

for all f, g, $h \in G$. Hence, λ defines a normalized 2-cocycle of G with coefficients in k^*. Since the A_g are uniquely determined up to scalar, another choice of these isomorphisms would replace the normalized 2-cocycle λ by $\lambda \cdot \mathrm{d}\mu$, the $\mu(g)$ are the mentioned scalars. Note that $\mathbb{P}(\mathcal{E}_\alpha \otimes \mathcal{E}_{\lambda^{-1}})$ has trivial Brauer class on the one hand, since the map $B_g \otimes A_g$ satisfies

$$(B(g) \otimes A_{\lambda^{-1}}(g)) \circ g^*(B(h) \otimes A_{\lambda^{-1}}(h)) = B(g * h) \otimes A_{\lambda^{-1}}(g * h)$$

for all g, $h \in G$. On the other hand, the Brauer class of $\mathbb{P}(\mathcal{E}_\alpha \otimes \mathcal{E}_{\lambda^{-1}})$ equals $\alpha \cdot I([\lambda^{-1}])$, which shoes $\alpha = I([\lambda])$. □

Theorem 3.12. *Let $f \colon G \to \mathrm{Pic}(X)$ be a crossed homomorphism such that its class is sent to zero by the map $H^1(G, \mathrm{Pic}(X)) \to H^3(G, k^*)$. Then one can find a G-action on $\mathbb{P}(\bigoplus_{g \in G} f(g))$, which commutes with the G-action on X such that the quotient $P := \mathbb{P}(\bigoplus_{g \in G} f(g))/G$ defines a Brauer–Severi variety over Y whose Brauer class is mapped to $[f]$ by Θ.*

Proof. The map $H^1(G, \mathrm{Pic}(X)) \to H^3(G, k^*)$ equals the composite of the two boundary maps

$$H^1(G, \mathrm{Pic}(X)) \to H^2(G, R_X^*/k^*) \to H^3(G, k^*),$$

where the first map arises from the short exact sequence $0 \to R^*/k^* \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0$, and the second map arises from the short exact sequence $0 \to k^* \to R_X^* \to R_X^*/k^* \to 0$, as was argued in the proof of [3, Lem. 4.2].

Let $f \colon G \to \mathrm{Pic}(X)$ be a crossed homomorphism. Choose divisors $D_g \in \mathrm{Div}(X)$ such that $\mathcal{O}_X(D_g)$ is isomorphic to $f(g)$. Since f is a crossed homomorphism, the divisors $D_g - D_{g*h} + g^*D_h$ are principal, i.e. of the form $\mathrm{div}(\overline{\psi(g,h)})$ for some rational function $\psi(g,h) \in R_X^*$. By definition, the first boundary map sends $[f]$ to $[\overline{\psi(g,h)}] \in H^2(G, R_X^*/k^*)$. Note that the rational function $\psi(g,h)$ defines an isomorphism $g^*f(h) \xrightarrow{\sim} f(g)^\vee \otimes f(g * h)$. Consider

$$\lambda(g,h,k) := g^*\psi(h,k)\psi(gh,k)^{-1}\psi(g,hk)\psi(g,h)^{-1},$$

and note that its class in R_X^*/k^* is trivial, since $\overline{\psi(g,h)}$ is a cocycle. By definition, the second boundary map sends $[\overline{\psi(g,h)}]$ to $[\lambda(g,h,k)]$.

Suppose now that the class of f is sent to zero by the composite of these two boundary maps. There hence exists a normalized 2-cocycle $\mu\colon G\times G \to\in k^*$ such that the differential of μ^{-1} equals λ. Replacing $\psi(g,h)$ by $\mu(g,h)\psi(g,h)$ yields isomorphisms

$$\alpha(g,h)\colon f(g)\otimes g^*f(h) \xrightarrow{\sim} f(g*h)$$

with the property that the composite

$$\begin{aligned} k^*g^*f(h) &\xrightarrow{k^*\alpha(g,h)} k^*f(g)^\vee\otimes k^*f(g*h) \xrightarrow{\alpha(k,g*h)} k^*f(g)^\vee\otimes f(k)^\vee\otimes f(k*g*h)\\ &\xrightarrow{(\alpha(k,g)^\vee)^{-1}} f(k*g)^\vee\otimes f(k*g*h)\end{aligned}$$

equals $\alpha(k*g,h)$. To ease the notation, set $\mathcal{E}_f := \bigoplus_{g\in G} f(g)$. For every $h\in G$, define the map

$$\beta_h := \bigoplus_{g\in G}(\alpha(h,g)\colon h^*f(g)\to f(h)^\vee\otimes f(h*g))\colon h^*\mathcal{E}_f\to f(h)^\vee\otimes\mathcal{E}_f.$$

By the above calculation, these maps have the property that the composite

$$k^*h^*\mathcal{E}_f \xrightarrow{k^*\beta_h} k^*f(h)^\vee\otimes k^*\mathcal{E}_f \xrightarrow{(\alpha(k,h)^\vee)^{-1}} f(h*k)^\vee\otimes\mathcal{E}_f$$

equals β_{k*h}. The universal property of Proj allows to define maps $\tau_h\colon \mathbb{P}(\mathcal{E}_f)\to\mathbb{P}(\mathcal{E}_f)$ via

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_f)}(-1)\otimes p^*f(h)\subset p^*\mathcal{E}_f\otimes p^*f(h)\xrightarrow{\beta_h^{-1}\otimes\mathrm{id}} p^*\mathcal{E}_f,$$

which satisfy $\tau_g\circ\tau_h=\tau_{gh}$ for all g, $h\in G$. The quotient $P:=\mathbb{P}(\mathcal{E}_f)/G\to Y$ clearly defines a Brauer–Severi variety over Y, whose Brauer class is thus mapped to $[f]$ via Θ by construction. □

Remark. In [18, Lem. 10], Martínez constructed a non-trivial Brauer–Severi variety over an Enriques surface Y as a quotient of $\mathbb{P}(\mathcal{O}_X\oplus\mathcal{L})$, where $\pi\colon X\to Y$ is the universal covering of Y and $\mathcal{L}\in\mathrm{Pic}(X)$ satisfies Beauville's condition, i.e. it is anti-invariant and $c_1^2(\mathcal{L})\equiv 2 \mod 4$. The quotient comes from the involution

$$\begin{array}{ccccccc} \mathbb{P}(\mathcal{O}_X\oplus\mathcal{L}) & \xrightarrow{\sim} & \mathbb{P}(\mathcal{L}^\vee\otimes(\mathcal{L}\oplus\mathcal{O}_X)) & \xrightarrow{\sim} & \mathbb{P}(\sigma^*\mathcal{O}_X\oplus\sigma^*\mathcal{L}) & \longrightarrow & \mathbb{P}(\mathcal{O}_X\oplus\mathcal{L})\\ & \searrow & & & \downarrow & & \downarrow p\\ & & & & X & \xrightarrow{\sigma} & X,\end{array}$$

where the square is a fiber product diagram and the first two maps in the upper row are induced by a fixed isomorphism $\sigma^*\mathcal{L}\cong\mathcal{L}^\vee$. Note that this construction agrees with our construction for the crossed homomorphism

$$\mathbb{Z}/2\mathbb{Z}\to\mathrm{Pic}(X),\ 1\mapsto\mathcal{L}.$$

Corollary 3.13. *The image of Θ equals the kernel of $H^1(G,\mathrm{Pic}(X))\to H^3(G,k^*)$.*

Proof. That the composite $\ker(\pi^{Br})\xrightarrow{\Theta} H^1(G,\mathrm{Pic}(X))\to H^3(G,k^*)$ is trivial follows from the fact that the maps

$$\tau_g=\begin{pmatrix} g^* & \varphi_g\\ 0 & \mathrm{id}\end{pmatrix}\colon \mathrm{Pic}(X)\oplus\mathbb{Z}\to\mathrm{Pic}(X)\oplus\mathbb{Z}$$

define a G-action, and thus $\varphi_{g*h}(1)=\varphi_g(1)\otimes g^*\varphi_h(1)$ for all g, $h\in G$.

Suppose that $[f]\in H^1(G,\mathrm{Pic}(X))$ is in the kernel of $H^1(G,\mathrm{Pic}(X))\to H^3(G,k^*)$. By the previous theorem, Θ sends the Brauer class of $\mathbb{P}(\bigoplus_{g\in G} f(g))/G$ to $[f]$, which shows that the kernel of $H^1(G,\mathrm{Pic}(X))\to H^3(G,k^*)$ equals the image of Θ. □

This discussion allows us to give a geometric interpretation of the seven-term sequence: it is given by

$$0 \to \ker(\pi^*) \to \mathrm{Pic}(Y) \xrightarrow{\pi^*} H^0(G, \mathrm{Pic}(X)) \to H^2(G, k^*) \xrightarrow{I} \ker(\pi^{Br}) \xrightarrow{\Theta} H^1(G, \mathrm{Pic}(X)) \\ \to H^3(G, k^*).$$

3.1 Application to cyclic étale coverings

Suppose that G is cyclic of order d. By theorem 2.4, the map $H^1(G, \mathrm{Pic}(X)) \to H^3(G, k^*)$ is given by the norm homomorphism. This section aims to describe this map in the two cases, where $\mathrm{Pic}(X)$ has no d-torsion or d is prime. Recall that the first group cohomology is given by

$$H^1(G, \mathrm{Pic}(X)) \cong \ker(N)/\mathrm{im}(\mathrm{id} - \sigma^*),$$

where $N\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$, $\mathcal{L} \mapsto \mathcal{L} \otimes \sigma^*\mathcal{L} \otimes \ldots \otimes (\sigma^{d-1})^*\mathcal{L}$, see [4, III. 1. Ex. 2]. It therefore suffices to compare the kernels of Nm and N.

Lemma 3.14. *Let G by cyclic, then $H^2(G, k^*) = 0$.*

Proof. By [4, III. 1. Ex. 2] $H^2(\mathbb{Z}/d\mathbb{Z}, k^*)$ is the quotient of the fixed points by the image of $z \mapsto z \cdot gz \cdots g^{d-1}z$, where g is a generator of $\mathbb{Z}/d\mathbb{Z}$. The action is trivial, and thus since k is algebraically closed, $H^2(\mathbb{Z}/d\mathbb{Z}, k^*) \cong k^*/\mathrm{im}(z \mapsto z^d) = 0$. □

Lemma 3.15. *Let $\pi\colon X \to Y$ be an étale covering of smooth projective varieties, and let $\mathcal{L} \in \mathrm{Pic}(X)$. Then the norm of $\mathcal{L}$ can be computed by*

$$\det \pi_*\mathcal{L} \cong \mathrm{Nm}(\mathcal{L}) \otimes \det \pi_*\mathcal{O}_X.$$

Proof. To see this choose an open cover $Y = \bigcup_i \mathrm{Spec}A_i$ such that $\mathcal{L}$ is trivial on $\pi^{-1}\mathrm{Spec}A_i$. We obtain trivialization maps $f_i\colon \mathcal{L}|_{\pi^{-1}\mathrm{Spec}A_i} \xrightarrow{\sim} \mathcal{O}_{\pi^{-1}\mathrm{Spec}A_i}$. Denote the resulting cocycle by $(f_{ij})_{ij}$ and denote by m_{ij} the $\mathcal{O}_Y$-linear endomorphism of $\pi_*\mathcal{O}_X$, induced by multiplication with f_{ij}. The norm of $\mathcal{L}$ is then defined by the cocycle $(\det(m_{ij}))_{ij}$. On the other hand, choose local trivializiation $g_i\colon \pi_*\mathcal{O}_X|_{\mathrm{Spec}A_i} \xrightarrow{\sim} \mathcal{O}_Y^{\oplus d}$ of $\pi_*\mathcal{O}_X$. The composition

$$\pi_*\mathcal{L}|_{\mathrm{Spec}A_i} \xrightarrow{\pi_* f_i} \pi_*\mathcal{O}_X|_{\mathrm{Spec}A_i} \xrightarrow{g_i} \mathcal{O}_Y^{\oplus d}$$

defines a trivialization of $\pi_*\mathcal{L}$, and hence $\det \pi_*\mathcal{L}$ is given by the cocycle

$$\det(g_j|_{U_{ij}} \circ (\pi_* f_j)|_{U_{ij}} \circ (\pi_* f_i^{-1})|_{U_{ij}} \circ g_i^{-1}|_{U_{ij}})_{ij} = \det(m_{ij})_{ij} \det(g_{ij})_{ij}.$$

Since $\det \pi_*\mathcal{O}_X$ is clearly given by the cocycle $(\det(g_{ij}))_{ij}$, the claim follows. □

Lemma 3.16. *Suppose that G is cyclic of order d. Then the kernel of $\pi^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$. Moreover,*

$$\pi_*\mathcal{O}_X \cong \bigoplus_{n=0}^{d-1} \mathcal{M}^n,$$

where $\mathcal{M} \in \ker(\pi^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X))$ is a generator.

Proof. Consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}/d\mathbb{Z}, H^q(X, \mathcal{O}_X^*)) \Rightarrow H^{p+q}(Y, \mathcal{O}_Y^*).$$

Since $E_2^{2,0} = 0$, we can identify the kernel of π^* with $E_\infty^{1,0} \cong E_2^{1,0} \cong \mathbb{Z}/d\mathbb{Z}$.

To prove the second part note that $\pi^*\pi_*\mathcal{O}_X \cong \bigoplus_n (\sigma^n)^*\mathcal{O}_X$, and that its descent is induced by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \mathrm{id} \\ \mathrm{id} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{id} & 0 \end{pmatrix} : \sigma^*\mathcal{O}_X^{\oplus d} \to \mathcal{O}_X^{\oplus d}.$$

After applying isomorphisms $(\sigma^n)^*\mathcal{O}_X \cong \mathcal{O}_X$ and a suitable base change we can identify this map with a diagonal matrix

$$\mathrm{diag}(\zeta_d^0 \cdot \sigma^\dagger, \dots, \zeta_d^{d-1} \cdot \sigma^\dagger) \colon \sigma^*\mathcal{O}_X^{\oplus d} \to \mathcal{O}_X^{\oplus},$$

where ζ_d is a primitive d-th root of unity. Denote by $\mathcal{M} \in \mathrm{Pic}(Y)$ the descent of $\zeta_d \cdot \sigma^\dagger \colon \sigma^*\mathcal{O}_X \to \mathcal{O}_X$. Then $\mathcal{M}$ generates $\ker \pi^*$, and we have

$$\pi_*\mathcal{O}_X \cong \bigoplus_{n=0}^{d-1} \mathcal{M}^n.$$

□

Lemma 3.17. *Suppose that G is cyclic of order d and denote by N the map*

$$N \colon \mathrm{Pic}(X) \to \mathrm{Pic}(X),\ \mathcal{L} \mapsto \mathcal{L} \otimes \sigma^*\mathcal{L} \otimes \ldots \otimes (\sigma^{d-1})^*\mathcal{L}.$$

Let $\mathcal{L} \in \ker(N)$ and suppose that the stabilizer $G_{\mathcal{L}}$ of $\mathcal{L}$ is trivial. Then $\det \pi_*\mathcal{L} \cong \mathcal{M}^{d/2}$ if d is even and $\det \pi_*\mathcal{L} \cong \mathcal{O}_Y$ if d is odd.

Proof. First, note that $\pi^*\pi_*\mathcal{L} \cong \bigoplus_i (\sigma^i)^*\mathcal{L}$. Indeed, since π^* is left adjoint to π_*, the isomorphism

$$\mathrm{Hom}(\pi^*\pi_*\mathcal{L}, (\sigma^i)^*\mathcal{L}) \cong \mathrm{Hom}(\pi_*\mathcal{L}, \pi_*(\sigma^i)^*\mathcal{L}) \cong \mathrm{Hom}(\pi_*\mathcal{L}, \pi_*\mathcal{L})$$

yields a non trivial map $\pi^*\pi_*\mathcal{L} \to (\sigma^i)^*\mathcal{L}$ associated to the identity $\pi_*\mathcal{L} \to \pi_*\mathcal{L}$ for all i. One can check locally that the direct sum of these maps $\pi^*\pi_*\mathcal{L} \to \bigoplus_i (\sigma^i)^*\mathcal{L}$ is an isomorphism.

Let $U \subset X$ be a small open subset on which $\mathcal{L}$ trivializes to $\mathcal{L}|_U \cong \mathcal{O}_U \cdot l$. Since $\mathcal{L}$ is in the kernel of N, the isomorphism $\pi^*\pi_*\mathcal{L} \cong \bigoplus_i (\sigma^i)^*\mathcal{L}$ implies

$$\pi^* \det \pi_*\mathcal{L} \cong \det \pi^*\pi_*\mathcal{L} \cong N(\mathcal{L}) \cong \mathcal{O}_X.$$

Observe that $\mathrm{Hom}((\sigma^i)^*\mathcal{L}, (\sigma^j)^*\mathcal{L})$ is non-trivial if and only if $i = j$, since the stabilizer of $\mathcal{L}$ is trivial. As a consequence, we see that $\pi_*\mathcal{L}$ is given by the descent of $\bigoplus_i (\sigma^i)^*\mathcal{L}$ via

$$A := \begin{pmatrix} 0 & 0 & \cdots & 0 & \lambda_0 \\ \lambda_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{d-1} & 0 \end{pmatrix} : \sigma^* \bigoplus_i (\sigma^i)^*\mathcal{L} \to \bigoplus_i (\sigma^i)^*\mathcal{L},$$

where $\lambda_0, \dots, \lambda_{d-1} \in k^*$. This map satisfies $(\sigma^{d-1})^*A \circ \cdots \circ A = \mathrm{id}$, and therefore the product $\lambda_0 \cdots \lambda_{d-1}$ equals one. The Laplace expansion along the first row shows that $\det A = (-1)^{d+1}$, and thus $\det \pi_*\mathcal{L}$ is isomorphic to the descent of $\mathcal{O}_X$ via $(-1)^{d+1} \cdot \sigma^\dagger \colon \sigma^*\mathcal{O}_X \to \mathcal{O}_X$, which is isomorphic to $\mathcal{M}^{d/2}$ if d is even, and is isomorphic to $\mathcal{O}_Y$ if d is odd. □

Lemma 3.18. *Suppose that G is cyclic and let $\mathcal{L} \in \mathrm{Pic}(X)$ with trivial stabilizer. Then the norm of $\mathcal{L}$ is trivial if and only if $N(\mathcal{L}) \cong \mathcal{O}_X$.*

Proof. Suppose that G has order d. By the previous lemma it suffices to show $\det \pi_*\mathcal{O}_X \cong \mathcal{M}^{d/2}$ if d is even and $\det \pi_*\mathcal{O}_X \cong \mathcal{O}_Y$ if d is odd. One calculates $\det \pi_*\mathcal{O}_X \cong \mathcal{M}^{\frac{d(d-1)}{2}}$, and since

$$\frac{d(d-1)}{2} \equiv d/2 \mod d \quad \text{if } d \text{ is even}$$
$$\frac{d(d-1)}{2} \equiv 0 \mod d \quad \text{if } d \text{ is odd,}$$

the assertion follows. □

Theorem 3.19. *Suppose that G is cyclic of order d and suppose that* $\mathrm{Pic}(X)[d] = 0$*, where* $[d]$ *denotes the kernel of multiplication by d. Then the map Θ is an isomorphism.*

Proof. Due to lemma 3.14 it suffices to show that Θ is surjective. Let $f\colon G \to \mathrm{Pic}(X)$ be a crossed homomorphism. It suffices to show that the norm of $\mathcal{L} := f(1)$ is trivial. Suppose that the stabilizer subgroup of $\mathcal{L}$ has order d/n, i.e. $(\sigma^n)^*\mathcal{L} \cong \mathcal{L}$. Since f is a crossed homomorphism, the line bundle $\mathcal{L}$ satisfies

$$(\mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{L})^{d/n} \cong \mathcal{L} \otimes \cdots \otimes (\sigma^{d-1})^*\mathcal{L} \cong \mathcal{O}_X.$$

We assumed that $\mathrm{Pic}(X)[d]$ is trivial, and hence we can fix an isomorphism

$$\varphi\colon \mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_X.$$

Choose an isomorphism $\psi\colon (\sigma^n)^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$, and consider the following diagram:

$$\begin{array}{ccc} \mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{L} & \xrightarrow{\quad\varphi\quad} & \mathcal{O}_X \\ \big\downarrow \bigotimes_i (\sigma^i)^*\psi & & \big\downarrow \\ (\sigma^n)^*(\mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{L}) & \xrightarrow{(\sigma^n)^*\varphi} & (\sigma^n)^*\mathcal{O}_X. \end{array}$$

This diagram commutes up to scalar. Since k is algebraically closed, we can multiply ψ by a scalar such that the diagram commutes. Denote by $\mathcal{L}_0$ the descent of $\mathcal{L}$ via ψ. We thus obtain a crossed homomorphism

$$\bar{f}\colon G/G_{\mathcal{L}} \to \mathrm{Pic}(X/G_{\mathcal{L}})$$

defined by assigning 1 to $\mathcal{L}_0$. Since $\mathcal{L}_0$ pulls back to $\mathcal{L}$, it suffices to show that the norm of $\mathcal{L}_0$ is trivial. Note that the stabilizer of $\mathcal{L}_0$ is trivial, and therefore the assertion follows from the previous lemma. □

Theorem 3.20. *Suppose that G is cyclic of prime order d. Then the sequence*

$$0 \to \ker(\pi^{Br}) \to H^1(G, \mathrm{Pic}(X)) \xrightarrow{\mathrm{Nm}} \ker(\pi^*) \cap \mathrm{im}([d]) \to 0$$

is exact.

Proof. First, we show that the norm homomorphism takes values in $\ker(\pi^*) \cap \mathrm{im}([d])$. Consider a crossed homomorphism $f\colon G \to \mathrm{Pic}(X)$, and define $\mathcal{L} := f(1)$. If the stabilizer of $\mathcal{L}$ is trivial, we already know that the norm of $\mathcal{L}$ is trivial as well. Suppose that the stabilizer is non-trivial and hence equal to G. Therefore, $\mathcal{L} \cong \pi^*\mathcal{L}_0$ for some $\mathcal{L}_0 \in \mathrm{Pic}(Y)$, and hence $\mathrm{Nm}(\mathcal{L}) \cong \mathcal{L}_0^d$. Since $\mathcal{L} \otimes \cdots \otimes (\sigma^{d-1})^*\mathcal{L}$ is trivial, the line bundle $\mathcal{L}_0^d$ is in the kernel of $\pi^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$.

To show that the sequence is surjective, observe that for a line bundle $\mathcal{L}_0^d \in \ker(\pi^*) \cap \mathrm{im}([d])$ the map $G \to \mathrm{Pic}(X)$, $n \mapsto \pi^*\mathcal{L}_0^n$ defines a crossed homomorphism with norm equal to $\mathcal{L}_0^d$. □

4 Application to Enriques surfaces

4.1 A construction by Kaji

In [15] Kaji constructs an Azumaya algebra from two torsion line bundles, which can be embedded as a conic bundle. We briefly recall this construction and show for Enriques surfaces that it only produces trivial Brauer–Severi varieties.

Let Y be a smooth projective variety over $\mathbb{C}$ and let $n \geq 2$. Kaji constructs a map $\mathcal{A}$ that fits into the commutative diagram

$$\begin{array}{ccccc} & & & & H^1(Y, \mathcal{O}_Y^*) \\ & & & & \downarrow c_1 \\ & & H^1(Y, \mathbb{Z}/n\mathbb{Z}) \times H^1(Y, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\smile} & H^2(Y, \mathbb{Z}/n\mathbb{Z}) \\ & & \downarrow \mathcal{A} & & \downarrow \\ H^1(Y, GL_n(\mathcal{O}_Y)) & \longrightarrow & H^1(Y, PGL_n(\mathcal{O}_Y)) & \longrightarrow & H^2(Y, \mathcal{O}_Y^*), \end{array}$$

where the lower row comes from the short exact sequence $0 \to \mathcal{O}_Y^* \to GL_n(\mathcal{O}_Y) \to PGL_n(\mathcal{O}_Y) \to 0$ and the right column is induced by the Kummer sequence. The map $\mathcal{A}$ is defined as follows: let $\mathcal{L}$ and $\mathcal{M}$ be n-torsion line bundles and fix isomorphisms $\varphi\colon \mathcal{O}_Y \to \mathcal{L}^n$ and $\psi\colon \mathcal{O}_Y \to \mathcal{M}^n$. Define

$$\mathcal{A}(\mathcal{L}, \mathcal{M}) := \bigoplus_{0 \leq i,j \leq n-1} \mathcal{L}^i \otimes \mathcal{M}^j.$$

This becomes an $\mathcal{O}_Y$-algebra by defining the graded multiplication

$$\mathcal{L}^i \otimes \mathcal{M}^j \otimes \mathcal{L}^k \otimes \mathcal{M}^l \xrightarrow{\zeta^{jk}} \mathcal{L}^i \otimes \mathcal{L}^k \otimes \mathcal{M}^j \otimes \mathcal{M}^l \to \mathcal{L}^r \otimes \mathcal{M}^s,$$

where ζ is a primitive n-th root of unity and $i + k \equiv r,\ j + l \equiv s \mod n$. The last map uses φ and ψ. Let $U \subset Y$ be an open affine subset, where $\mathcal{L}$ has a local generator l and $\mathcal{M}$ has a local generator m. Note that l^n and $\varphi(1)|_U$ generate $\mathcal{L}^n$, and thus there is a local unit $a \in \Gamma(U, \mathcal{O}_U)^*$ such that

$$a\varphi(1)|_U = l^n.$$

Similarly, there is a local unit $b \in \Gamma(U, \mathcal{O}_U)^*$ such that

$$b\psi(1)|_U = m^n.$$

Consequently, $\mathcal{A}|_U$ is generated as an $\mathcal{O}_U$-algebra by l and m that satisfy the relations

$$l^n = a, \quad m^n = b, \quad lm = \zeta ml.$$

In the case $n = 2$, this Azumaya algebra corresponds to the Brauer–Severi variety defined as the conic bundle $V(q) \subset \mathbb{P}(\mathcal{E})$, where $\mathcal{E} := \mathcal{O}_Y \oplus \mathcal{L} \oplus \mathcal{M}$ and

$$q := \frac{1}{1 \otimes 1} - \frac{1}{\varphi(1)} - \frac{1}{\psi(1)} \in H^0(Y, \mathrm{Sym}^2 \mathcal{E}^\vee) \cong H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)).$$

Note that this construction is trivial for an Enriques surface Y with $n = 2$: Let α be the generator of $H^1(Y, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and let $\pi\colon X \to Y$ be the universal covering. Then

$$\pi^*(\alpha \smile \alpha) = \pi^*\alpha \smile \pi^*\alpha = 0,$$

since $\pi^*\alpha \in H^1(X, \mathbb{Z}/2\mathbb{Z}) = 0$. Hence, $\alpha \smile \alpha$ is either 0 or k_Y by [3, Prop. 3.5], and therefore $\mathcal{A}(\alpha, \alpha)$ is trivial, since $\alpha \smile \alpha$ is in the image of c_1.

4.2 Conic bundles

From now on let Y be an Enriques surface with universal covering $\pi\colon X \to Y$ such that π^{Br} is trivial. Let $\mathcal{L} \in \operatorname{Pic}(X)$ be an anti-invariant line bundle and fix an isomorphism $\psi\colon \sigma^*\mathcal{L} \to \mathcal{L}^\vee$. Then theorem 3.12 constructs an involution on $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$ that is given by

$$\begin{array}{ccccccc}
\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) & \xrightarrow{\sim} & \mathbb{P}(\mathcal{L}^\vee \otimes (\mathcal{L} \oplus \mathcal{O}_X)) & \xrightarrow{\sim} & \mathbb{P}(\sigma^*\mathcal{O}_X \oplus \sigma^*\mathcal{L}) & \longrightarrow & \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) \\
 & \searrow & & & \downarrow & & \downarrow p \\
 & & & & X & \xrightarrow{\sigma} & X,
\end{array}$$

where the square is a fiber product diagram and the first two maps in the upper row are induced by ψ. Denote the composition of the upper row by τ and the quotient $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})/\tau$ by P. Recall that $P \to Y$ defines a non-trivial Brauer–Sever variety if $[\mathcal{L}] \in H^1(\mathbb{Z}/2\mathbb{Z}, \operatorname{Pic}(X))$ is non-trivial.

Definition 4.1. The vector bundle $\mathcal{F}_\mathcal{L}$ is defined as the descent of $\mathcal{L}^\vee \oplus \mathcal{L}$ via the isomorphism

$$\varphi := \begin{pmatrix} 0 & \psi \\ \sigma^*\psi^{-1} & 0 \end{pmatrix} : \sigma^*(\mathcal{L}^\vee \oplus \mathcal{L}) \to \mathcal{L}^\vee \oplus \mathcal{L}.$$

Observe that the Veronese embedding

$$i\colon \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) \hookrightarrow \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^2) \cong \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L})$$

fits into the commutative diagram

$$\begin{array}{ccc}
\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) & \xrightarrow{\tau} & \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) \\
\downarrow i & & \downarrow i \\
\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L}) & \longrightarrow & \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L}),
\end{array}$$

where the lower map denotes the involution which induces the descent to $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{F}_\mathcal{L})$. Hence, i descends to an embedding $\iota\colon P \hookrightarrow \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{F}_\mathcal{L})$ that embeds P as a conic bundle.

Lemma 4.2. *There is an isomorphism* $\mathcal{F}_\mathcal{L} \cong \pi_*\mathcal{L}$.

Proof. Note that both bundles pull back to $\mathcal{L}^\vee \oplus \mathcal{L}$. Let $f\colon \sigma^*(\mathcal{L}^\vee \oplus \mathcal{L}) \to \mathcal{L}^\vee \oplus \mathcal{L})$ be the isomorphisms that induce the descent to $\pi_*\mathcal{L}$. We can write

$$f = \begin{pmatrix} 0 & \lambda\psi \\ \mu\sigma^*\psi^{-1} & 0 \end{pmatrix}$$

for some scalars λ, $\mu \in \mathbb{C}^*$. Since f satisfies $f \circ \sigma^* f = \mathrm{id}$, we conclude $\lambda\mu = 1$. Define

$$g := \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} : \mathcal{L}^\vee \oplus \mathcal{L} \to \mathcal{L}^\vee \oplus \mathcal{L},$$

and note that the following diagram commutes:

$$\begin{array}{ccc}
\sigma^*(\mathcal{L}^\vee \oplus \mathcal{L}) & \xrightarrow{f} & \mathcal{L}^\vee \oplus \mathcal{L} \\
\downarrow \sigma^* g & & \downarrow g \\
\sigma^*(\mathcal{L}^\vee \oplus \mathcal{L}) & \xrightarrow{\varphi} & \mathcal{L}^\vee \oplus \mathcal{L}.
\end{array}$$

Therefore, the map g descends to an isomorphism $\overline{g}\colon \pi_*\mathcal{L} \xrightarrow{\sim} \mathcal{F}_\mathcal{L}$. $\square$

Corollary 4.3. *By [22, Prop. 1.2] the vector bundle $\mathcal{F}_{\mathcal{L}}$ is special, i.e. simple and invariant under tensoring with ω_Y, and stable.*

Lemma 4.4. $H^0(X, \mathcal{L}^i) = 0 = H^2(X, \mathcal{L}^i)$ *for all* $i \neq 0$.

Proof. Suppose there is a non-zero global section $s \in H^0(X, \mathcal{L}^i)$. Then $t := H^0(\psi^i)(\sigma^* s)$ defines a non-zero global section of $\mathcal{L}^{-i}$, which implies $\mathcal{L}^i \cong \mathcal{O}_X$, and hence $i = 0$. This proves the first equality. The second equality follows from Serre duality. □

Lemma 4.5. *There is an isomorphism* $H^0(\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L}), \mathcal{O}_{\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L})}(2)) \cong \mathbb{C}^{\oplus 2}$, *obtained from the push forward on* X.

Proof. We calculate

$$\begin{aligned} H^0(\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L}), \mathcal{O}_{\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L})}(2)) &\cong H^0(X, \mathrm{Sym}^2(\mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^\vee)) \\ &\cong H^0(X, \mathcal{O}_X \oplus \mathcal{L}^2 \oplus \mathcal{L}^{-2} \oplus \mathcal{L} \oplus \mathcal{L}^\vee \oplus \mathcal{O}_X) \\ &\cong \mathbb{C}^{\oplus 2}, \end{aligned}$$

where the last isomorphism uses the previous lemma. □

Lemma 4.6. *The image of* $\iota \colon P \hookrightarrow \mathbb{P}(\mathcal{O}_Y \oplus \pi_* \mathcal{L})$ *equals* $V(1, -1) \subset \mathbb{P}(\mathcal{O}_Y \oplus \pi_* \mathcal{L})$, *where* $(1, -1)$ *denotes a global section of* $\mathcal{O}_{\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L})}(2)$ *via the previous isomorphism.*

Proof. Observe that the Veronese embedding sends locally $[u : v]$ to $[u^2 : uv : v^2]$, which satisfies the equation $x_1^2 - x_0 x_2$. Moreover, note that the term x_1^2 corresponds to the global section 1 from the first copy of $\mathcal{O}_X$ and that the term $x_0 x_2$ corresponds to the global section -1 from the second copy of $\mathcal{O}_X$. Therefore, the conic P satisfies the equation $(1, -1) = 0$. □

Lemma 4.7. $\psi \otimes \sigma^* \psi^{-1} = \sigma^\dagger$, *where* $\sigma^\dagger \colon \sigma^* \mathcal{O}_X \to \mathcal{O}_X$ *is the map of sheaves of rings induced by* σ.

Proof. Let $U \subset X$ be an open subset. Write $\mathcal{L}|_U = \mathcal{O}_U \cdot l$. There is an invertible local section $\lambda \in \Gamma(U, \mathcal{O}_X)^*$ such that the map ψ is given by $\sigma^* l \mapsto \lambda \cdot \frac{1}{l}$. Then $\psi \otimes (\sigma^* \psi^{-1})$ is given by

$$\sigma^* l \otimes \frac{1}{\sigma^* l} \mapsto \lambda \cdot \frac{1}{l} \otimes \frac{1}{\lambda} \cdot l = \frac{1}{l} \otimes l.$$

□

Lemma 4.8. *The composite*

$$H^0(S, \mathrm{Sym}^2(\mathcal{O}_Y \oplus \pi_* \mathcal{L}^\vee)) \to H^0(X, \mathrm{Sym}^2(\mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^\vee)) \cong \mathbb{C}^{\oplus 2}$$

is an isomorphism.

Proof. The map $\mathrm{Sym}^2(\sigma^\dagger \oplus \varphi_{\mathcal{L}})$ can be written as $\mathrm{Sym}^2(\sigma^\dagger \oplus \varphi_{\mathcal{L}}) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where

$$A := \begin{pmatrix} \sigma^\dagger & 0 & 0 \\ 0 & 0 & (\sigma^* \psi^{-1})^{\otimes 2} \\ 0 & \psi^{\otimes 2} & 0 \end{pmatrix}, \quad B := \begin{pmatrix} \sigma^\dagger & 0 & 0 \\ 0 & 0 & \sigma^* \psi^{-1} \\ 0 & \psi & 0 \end{pmatrix},$$

due to lemma 4.7. This implies that the involution

$$H^0(X, \mathrm{Sym}^2(\mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^\vee)) \overset{\sigma^*}{\to} H^0(X, \sigma^* \mathrm{Sym}^2(\mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^\vee)) \overset{\mathrm{Sym}^2 \varphi}{\to} H^0(X, \mathrm{Sym}^2(\mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^\vee))$$

can be identified with

$$H^0(X, \mathcal{O}_X^{\oplus 2}) \overset{\sigma^*}{\to} H^0(X, \sigma^* \mathcal{O}_X^{\oplus 2}) \overset{\sigma^\dagger \oplus \sigma^\dagger}{\to} H^0(X, \mathcal{O}_X^{\oplus 2}),$$

which is trivial. □

Lemma 4.9. *All smooth conic bundles $C \in |\mathcal{O}_{\mathbb{P}(\mathcal{O}_Y \oplus \pi_*\mathcal{L})}(2)|$ are isomorphic to each other.*

Proof. Let C be given by $C = V(a,b)$. Local computations show that C is smooth if and only if $a \neq 0 \neq b$. Choose square roots $\sqrt{a}$, $\sqrt{b} \in \mathbb{C}$, and define the isomorphism

$$h := \begin{pmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & \sqrt{b} \end{pmatrix} : \mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^\vee \to \mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^\vee.$$

Note that $h \circ \varphi = \varphi \circ \sigma^* h$, and thus h descends to an isomorphism $\overline{h}\colon \mathcal{O}_Y \oplus \pi_*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_Y \oplus \pi_*\mathcal{L}$. Observe that the induced automorphism on $\mathbb{P}(\mathcal{O}_Y \oplus \pi_*\mathcal{L})$ sends $V(1,1)$ to C. □

Proposition 4.10. *Let $C \in |\mathcal{O}_{\mathbb{P}(\mathcal{O}_Y \oplus \pi_*\mathcal{L})}(2)|$ be a smooth conic bundle. Then $C \to Y$ defines a non-trivial Brauer–Severi variety if and only if $[\mathcal{L}] \in H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X))$ is non-trivial.*

Proof. We already know that the Brauer class of P is non-trivial if $[\mathcal{L}] \neq 0$ in $H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X))$. Suppose the converse, i.e. there is a line bundle $\mathcal{L}_1 \in \mathrm{Pic}(X)$ such that $\mathcal{L} \cong \mathcal{L}_1 \otimes \sigma^*\mathcal{L}_1^\vee$. Observe that

$$\pi^*(\mathrm{Nm}(\mathcal{L}_1) \otimes (\mathcal{O}_Y \oplus \pi_*\mathcal{L})) \cong \mathrm{Sym}^2(\mathcal{L}_1 \oplus \sigma^*\mathcal{L}_1).$$

Moreover, both bundles come with isomorphic descent datum. Therefore, tensoring $\mathcal{O}_Y \oplus \pi_*\mathcal{L}$ with $\mathrm{Nm}(\mathcal{L}_1)$ shows that P is isomorphic to $\mathbb{P}(\mathcal{L}_1)$, and hence trivial. □

Lemma 4.11. *The corresponding Azumaya algebra is of the form $\mathcal{O}_Y \oplus \pi_*\mathcal{L} \oplus \omega_Y$.*

Proof. By construction the Azumaya algebra is the descent of $\mathcal{E}nd(\mathcal{O}_X \oplus \mathcal{L})$ via

$$\begin{aligned} \mathcal{E}nd(\sigma^*\mathcal{O}_X \oplus \sigma^*\mathcal{L}) &\to \mathcal{E}nd(\mathcal{L}^\vee \otimes (\mathcal{O}_X \oplus \mathcal{L})) \cong \mathcal{E}nd(\mathcal{O}_X \oplus \mathcal{L}) \\ f &\mapsto (\varphi) \circ f \circ \varphi^{-1}. \end{aligned}$$

Using the isomorphism $\mathcal{E}nd(\mathcal{O}_X \oplus \mathcal{L}) \cong \mathcal{O}_X \oplus \mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L}$, local computations allow to identify the above map with

$$\begin{pmatrix} 0 & \sigma^\dagger & 0 & 0 \\ \sigma^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & \psi \\ 0 & 0 & \sigma^*\psi^{-1} & 0 \end{pmatrix} : \sigma^*(\mathcal{O}_X \oplus \mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L}) \to \mathcal{O}_X \oplus \mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L}.$$

After a suitable base change on $\mathcal{O}_X^{\oplus 2}$, this map can be identified with $\sigma^\dagger \oplus (-\sigma^\dagger) \oplus \varphi_{\mathcal{L}}$, which proves the claim. □

Proposition 4.12. *Let $\mathcal{M} \in \mathrm{Pic}(Y)$ be a line bundle and let $\mathcal{F}$ be a stable vector bundle over Y of rank 2 for a fixed polarization $\mathcal{O}_Y(1) \in \mathrm{Pic}(Y)$. Suppose there is a conic bundle $C \hookrightarrow \mathbb{P}(\mathcal{M} \oplus \mathcal{F})$, such that all fibers of $C \to Y$ are smooth. Then there is a line bundle $\mathcal{N}$ such that*

$$\mathcal{N}^\vee \otimes (\mathcal{M} \oplus \mathcal{F}) \cong \mathcal{O}_Y \oplus \pi_*\mathcal{L},$$

where $\mathcal{L} \in \mathrm{Pic}(X)$ is an anti-invariant line bundle. Moreover, the corresponding isomorphism $\mathbb{P}(\mathcal{M} \oplus \mathcal{F}) \xrightarrow{\sim} \mathbb{P}(\mathcal{O}_Y \oplus \pi_*\mathcal{L})$ sends C to the previously constructed conic bundle.

Proof. Suppose that C is given by a global section of $\mathcal{O}_{\mathbb{P}(\mathcal{M}\oplus\mathcal{F})}(2) \otimes p^*\mathcal{N}_0$ for some $\mathcal{N}_0 \in \mathrm{Pic}(Y)$. Since all fibers are smooth, the associated bilinear form is non-degenerate and hence we obtain an isomorphism

$$\mathcal{M} \oplus \mathcal{F} \to \mathcal{N}_0 \otimes (\mathcal{M}^\vee \oplus \mathcal{F}^\vee).$$

Taking determinants reveals

$$(\mathcal{M}\otimes\det\mathcal{F})^2\cong\mathcal{N}_0^3.$$

Define $\mathcal{N}_1 := \mathcal{N}_0^\vee\otimes\mathcal{M}\otimes\det\mathcal{F}$, which thus has the properties

$$\mathcal{N}_1^2\cong\mathcal{N}_0,\quad \mathcal{N}_1^3\cong\mathcal{N}\otimes\det\mathcal{F}.$$

Tensoring the isomorphism $\mathcal{M}\oplus\mathcal{F}\to\mathcal{N}_0\otimes(\mathcal{M}^\vee\oplus\mathcal{F}^\vee)$ with $\mathcal{N}_1^\vee$ results in

$$\varphi=\begin{pmatrix} f & g\\ g^\vee & h\end{pmatrix}:\mathcal{N}_1^\vee\otimes(\mathcal{M}\oplus\mathcal{F})\to\mathcal{N}_1\otimes(\mathcal{M}^\vee\oplus\mathcal{F}^\vee)$$

with inverse

$$\varphi^{-1}=\begin{pmatrix} f' & g'\\ g'^\vee & h'\end{pmatrix}:\mathcal{N}_1\otimes(\mathcal{M}^\vee\oplus\mathcal{F}^\vee)\to\mathcal{N}_1^\vee\otimes(\mathcal{M}\oplus\mathcal{F}).$$

We first claim that f and h are isomorphisms. Suppose that f is not an isomorphism, and thus not surjective. Then g cannot be zero. The stability of $\mathcal{F}$ implies the inequality of reduced Hilbert polynomials $p(\mathcal{F})<p(\mathcal{N}_1^2\otimes\mathcal{M}^\vee)$. On the other hand, the composition $\varphi\circ\varphi^{-1}$ equals the identity, and hence

$$\mathrm{id}=f\circ f'+g\circ g'^\vee.$$

Since f is not surjective, the composite $f\circ f'$ cannot be surjective either. Therefore, $g'^\vee$ is non-zero and hence injective. The stability of $\mathcal{F}$ implies $p(\mathcal{N}_1^2\otimes\mathcal{M}^\vee)<p(\mathcal{F})$, which is a contradiction. This shows that f and f' are isomorphisms. A very similar argument shows that h and h' are isomorphisms.

Observe that

$$\mathcal{F}^\vee\cong\det\mathcal{F}^\vee\otimes\mathcal{F}\cong\mathcal{N}_1^{-3}\otimes\mathcal{M}\otimes\mathcal{F},$$

where the last isomorphism comes from $\mathcal{N}_1^3\cong\mathcal{M}\otimes\det\mathcal{F}$. Since f defines an isomorphism between $\mathcal{N}_1^\vee\otimes\mathcal{M}$ and its dual, the tensor product $\mathcal{N}_1^\vee\otimes\mathcal{M}$ is either isomorphic to $\mathcal{O}_Y$ or to ω_Y. Therefore, tensoring h with $\mathrm{id}_{\mathcal{N}_1}$ yields a symmetric isomorphism

$$\mathcal{F}\to\mathcal{F},\quad \text{resp. } \mathcal{F}\to\omega_Y\otimes\mathcal{F}.$$

If we are in the first situation, replacing $\mathcal{M}\oplus\mathcal{F}$ by $\omega_S\otimes\mathcal{M}\oplus\omega_S\otimes\mathcal{F}$ still allows us to define the same non-degenerate symmetric bilinear form. The argument above holds as well. However, we would then be in the situation that h defines a symmetric isomorphism $\mathcal{F}\to\omega_Y\otimes\mathcal{F}$. Since $\mathcal{F}$ is simple and invariant under tensoring with ω_Y, we can assume $\mathcal{F}\cong\pi_*\mathcal{L}_0$ for some line bundle $\mathcal{L}_0$ on X by [22, Prop. 1.2]. Note that

$$\mathcal{O}_X\cong\pi^*\det(\mathcal{N}_1^\vee\otimes\mathcal{F})\cong\pi^*\mathcal{N}_1^{-2}\otimes\mathcal{L}_0\otimes\sigma^*\mathcal{L}_0,$$

and define $\mathcal{L}:=\pi^*\mathcal{N}_1^\vee\otimes\mathcal{L}_0$. This is anti-invariant. Moreover, $\pi_*\mathcal{L}\cong\mathcal{N}_1^\vee\otimes\mathcal{F}$, since the pullback of both bundles to X comes with the same descent datum. Therefore, the composite

$$\mathbb{P}(\mathcal{M}\oplus\mathcal{F})\xrightarrow{\sim}\mathbb{P}(\omega_Y\oplus\pi_*\mathcal{L})\xrightarrow{\sim}\mathbb{P}(\mathcal{O}_Y\oplus\pi_*\mathcal{L})$$

sends C to a conic bundle defined by a global section of $\mathcal{O}_{\mathbb{P}(\mathcal{O}_Y\oplus\pi_*\mathcal{L})}(2)$. This composition equals the isomorphism obtained from tensoring $\mathcal{M}\oplus\mathcal{F}$ with $\mathcal{N}:=\mathcal{N}_1$, resp. $\mathcal{N}:=\mathcal{N}_1\otimes\omega_Y$.

Note that $\mathcal{L}_0$ is not trivial, since otherwise $\pi_*\mathcal{L}_0\cong\mathcal{O}_Y\oplus\omega_Y$, which is not stable. Hence, $\mathcal{L}$ is non-trivial as well. By lemma 4.9 all smooth conic bundles in $|\mathcal{O}_{\mathbb{P}(\mathcal{O}_Y\oplus\pi_*\mathcal{L})}(2)|$ are isomorphic, and hence the isomorphism $\mathbb{P}(\mathcal{M}\oplus\mathcal{F})\xrightarrow{\sim}\mathbb{P}(\mathcal{O}_Y\oplus\pi_*\mathcal{L})$ sends C to the conic bundle constructed previously. □

4.3 Application to moduli spaces of vector bundles on Enriques surfaces

We fix a polarization $\mathcal{O}_Y(1) \in \text{Pic}(Y)$ on Y and denote the induced polarization on X by $\mathcal{O}_X(1) := \pi^*\mathcal{O}_Y(1)$. In this section we suppose that Y is unnodal, i.e. there are no (-2)-curves on Y.

Recall that $\pi_*\mathcal{L}$ is stable, satisfies $\det \pi_*\mathcal{L} \cong \omega_Y$ and $c_2(\pi_*\mathcal{L}) = -\frac{1}{2}c_1^2(\mathcal{L}) = 1+2n$ for some $n \geq 1$. Therefore, $\pi_*\mathcal{L}$ defines a point in $M_Y(v_n, \omega_Y)$, the moduli space of semistable vector bundles on Y, where $v_n := (2, 0, -2n)$.

Note that by [14, Cor. 4.6.7] $M_Y(v_n, \omega_Y) = M_Y(v_n, \omega_Y)^s$, and the dimension equals $8n+1$. The moduli space is normal with torsion canonical divisor by [23, Thm. 7.1]. Moreover, $M_Y(v_n, \omega_Y)$ is singular at $\mathcal{F} \in M_Y(v_n, \omega_Y)$ if and only if $\mathcal{F} \cong \omega_Y \otimes \mathcal{F}$, [16].

Lemma 4.13. *Let $\mathcal{F}$ be rank 2 vector bundle with $\det \mathcal{F} \cong \omega_Y$ and $c_2(\mathcal{F}) = 1+2n$. Then the following hold true:*

(i) $\mathcal{F}^\vee \cong \omega_Y \otimes \mathcal{F}$.

(ii) $\mathcal{F} \in M_Y(v_n, \omega_Y)$ *if and only if* $\mathcal{F}^\vee \in M_Y(v_n, \omega_Y)$.

(iii) *If* $\mathcal{F} \in M_Y(v_n, \omega_Y)$ *then* $H^0(Y, \mathcal{F}) = 0$.

Moreover, the following conditions are equivalent

(iv) $\omega_Y \otimes \mathcal{F} \cong \mathcal{F}$.

(v) $\mathcal{F} \cong \mathcal{F}^\vee$.

(vi) $\text{Sym}^2\mathcal{F}$ *has a non-vanishing global section.*

Proof. Tensoring the split exact sequence

$$0 \to \omega_Y \to \mathcal{F} \otimes \mathcal{F} \to \text{Sym}^2\mathcal{F} \to 0$$

with ω_Y induces a non-vanishing global section on $\omega_Y \otimes \mathcal{F} \otimes \mathcal{F}$, which is equivalent to giving an isomorphism $\omega_Y \otimes \mathcal{F} \xrightarrow{\sim} \mathcal{F}^\vee$.

Note that tensoring with ω_Y preserves stability since it does not affect the reduced Hilbert polynomial. This proves (ii).

Any non-trivial global section of $\mathcal{F}$ induces a map $\mathcal{O}_Y \to \mathcal{F}$, which is injective since $\mathcal{F}$ is torsion-free. The reduced Hilbert polynomial of $\mathcal{O}_Y$ equals $p(\mathcal{O}_Y) = \frac{1}{2}m^2 + \frac{1}{d}$ and the reduced Hilbert polynomial of $\mathcal{F}$ equals $p(\mathcal{F}) = \frac{1}{2}m^2 + \frac{1}{4d}c_1^2(\mathcal{L}) + \frac{1}{d}$, where d is defined as $d := c_1^2(\mathcal{O}_Y(1))$. However, $c_1^2(\mathcal{L}) < 0$, which implies $p(\mathcal{O}_Y) > p(\mathcal{F})$. This contradicts the stability of $\mathcal{F}$.

The equivalence of (i) and (ii) follows from the first assertion. To show the equivalence of (ii) and (iii) note that the split exact sequence

$$0 \to \omega_Y \to \mathcal{F} \otimes \mathcal{F} \to \text{Sym}^2\mathcal{F} \to 0$$

implies that $\text{Sym}^2\mathcal{F}$ has a non-vanishing global section if and only if $\mathcal{F} \otimes \mathcal{F}$ has a non-vanishing global section. The latter is equivalent to giving an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{F}^\vee$. □

Theorem 4.14. *The map π^{Br} is injective if and only if $M_Y(v_n, \omega_Y)$ is smooth for all $n \geq 1$.*

Proof. By [16] the moduli space $M_Y(v_n, \omega_Y)$ has a singularity at $\mathcal{F}$ if and only if $\mathcal{F}$ is invariant under tensoring with ω_Y. By [22, Prop. 1.2], the pullback of $\mathcal{F}$ to X splits as $\pi^*\mathcal{F} \cong \mathcal{L} \oplus \sigma^*\mathcal{L}$. Since $\det \mathcal{F} \cong \omega_Y$, the line bundle $\mathcal{L}$ is anti-invariant, and since $c_2(\mathcal{F}) = 1+2n \equiv 1 \mod 2$, the line bundle $\mathcal{L}$ satisfies $c_1^2(\mathcal{L}) \equiv 2 \mod 4$. By theorem 2.12, this equivalent to the triviality of π^{Br}.

On the other hand, if π^{Br} is trivial, then there exists an anti-invariant line bundle $\mathcal{L}$ on X by, theorem 2.12, such that $c_1^2(\mathcal{L}) = -2-4n \equiv 2 \mod 4$. Therefore, the push forward $\pi_*\mathcal{L}$ defines a singularity in $M_Y(v_n, \omega_Y)$. □

5 Application to bielliptic surfaces

In [6] Ferrari, Tiribassi, and Vodrup apply Beauville's theorem to bielliptic surfaces and their canonical coverings. We will give a short introduction to bielliptic surfaces and prove the main results of [6] concerning the Brauer map of the canonical covering of bielliptic surfaces of type 1, 3, and 5 by using theorem 3.20. Furthermore, explicit examples will be presented. For the case of bielliptic surfaces of type 2, we refer to [6, Thm. 5.21].

Definition 5.1. [1, 10.20] A bielliptic surface is a minimal smooth projective surface S over $\mathbb{C}$ with Kodaira dimension $\kappa(S) = 0$, irregularity $q(S) = 1$, and geometric genus $p_g(S) = 0$.

Bielliptic surfaces can be classified into seven different types, due to the following theorem:

Theorem 5.2. *[1, Thm. 10.25] Let S be a bielliptic surface. Then there are two elliptic curves A, B such that $S \cong (A \times B)/G$, where G is a finite group that acts on A by translation and on B by automorphisms. The quotient A/G is another elliptic curve whereas the quotient B/G is isomorphic to $\mathbb{P}^1$.*

We then obtain seven different types due to Bagnera and DeFranchis, by [1, List 10.27], [6, Table 1]:

Type	G	Order of ω_S	Brauer group
1	$\mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z}$
3	$\mathbb{Z}/4\mathbb{Z}$	4	$\mathbb{Z}/2\mathbb{Z}$
4	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	4	0
5	$\mathbb{Z}/3\mathbb{Z}$	3	$\mathbb{Z}/3\mathbb{Z}$
6	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	3	0
7	$\mathbb{Z}/6\mathbb{Z}$	6	0

Denote by $0_A \in A$ and $0_B \in B$ the respective identity points, and by $T_a\colon A \to A$ the translation by $a \in A$. The map

$$A \to \mathrm{Pic}^0(A),\ a \mapsto P_a := T_a^*\mathcal{O}_A(0_A) \otimes \mathcal{O}_A(-0_A) \cong \mathcal{O}_A(a - 0_A)$$

is an isomorphism, [21, II. 6. Cor. 4, 8. Thm. 1].

Definition 5.3. [21, II. 8] The normalized Poincaré bundle is defined as

$$\mathcal{P}_A := \mathcal{O}_{A\times A}(\Delta_A) \otimes pr_1^*\mathcal{O}_A(-0_A) \otimes pr_2^*\mathcal{O}_A(-0_A).$$

Consider the map

$$\begin{aligned} L\colon \mathrm{Pic}(A) \times \mathrm{Pic}(B) \times \mathrm{Hom}(B, A) &\to \mathrm{Pic}(A \times B) \\ (\mathcal{L}_A, \mathcal{L}_B, \varphi) &\mapsto (\mathrm{id}_A \times \varphi)^*\mathcal{P}_A \otimes pr_A^*\mathcal{L}_A \otimes pr_B^*\mathcal{L}_B, \end{aligned}$$

and note that this is an isomorphism. Indeed, to prove the surjectivity consider a line bundle $\mathcal{L} \in \mathrm{Pic}(A \times B)$. Define $\mathcal{L}_A := \mathcal{L}|_{A\times\{0_B\}}$, $\mathcal{L}_B := \mathcal{L}|_{\{0_A\}\times B}$, and $\mathcal{L}_0 := \mathcal{L} \otimes pr_A^*\mathcal{L}_A^\vee \otimes pr_B^*\mathcal{L}_B^\vee$. By the universal property of the Poincaré bundle there is a morphism $\varphi\colon B \to A$ such that $\mathcal{L}_0$ is isomorphic to $(\mathrm{id} \times \varphi)^*\mathcal{P}_A$, [21, II. 8]. Therefore, $\mathcal{L} \cong L(\mathcal{L}_A, \mathcal{L}_B, \varphi)$. The injectivity is a consequence of the seesaw theorem, [21, II. 5. Cor. 6].

Lemma 5.4. *Let $\mu \in A$ be an n-torsion point and let $\xi\colon B \to B$ be an automorphism of order n. Define $\sigma\colon A \times B \to A \times B,\ (a, b) \mapsto (a + \mu, \xi(b))$. Then*

$$\sigma^*L(\mathcal{M}_A, \mathcal{M}_B, \varphi) \cong L(T_\mu^*\mathcal{M}_A, \xi^*\mathcal{M}_B \otimes \xi^*\varphi^*P_\mu, \varphi \circ \xi).$$

Proof. The effects on $\mathcal{M}_A$ and $\mathcal{M}_B$ are obvious. Since

$$(T_\mu \times \xi)^* L(0,0,\varphi) \cong (T_\mu \times \xi)^* (\mathrm{id} \times \varphi)^* \mathcal{P}_A \cong (\mathrm{id} \times \varphi \circ \xi)^* (T_\mu^* \times \mathrm{id})^* \mathcal{P}_A,$$

it suffices to show $(T_\mu^* \times \mathrm{id})^* \mathcal{P}_A \cong \mathcal{P}_A \otimes pr_2^* P_\mu$, which is a straight forward calculation. □

Lemma 5.5. *Let $\mu \in A$ be an n-torsion point. Then the diagram*

$$\begin{array}{ccc} A & \xrightarrow{a \mapsto P_a} & \mathrm{Pic}^0(A) \\ \downarrow q & & \downarrow \mathrm{Nm} \\ A/\mu & \xrightarrow{\overline{a} \mapsto P_{\overline{a}}} & \mathrm{Pic}^0(A/\mu) \end{array}$$

commutes.

Proof. Let $a \in A$. It suffices to show $\mathrm{Nm}(\mathcal{O}_A(-a)) \cong \mathcal{O}_{A/\mu}(-\overline{a})$. Let $U \subset A$ be a small open subset containing a. Then there is a local section $f \in \Gamma(U, \mathcal{O}_U)$ such that $\mathcal{O}_A(-a)|_U$ is generated by f. Consequently, f has a zero of order 1 over a and is non-zero everywhere else. Since $\mathrm{Nm}(\mathcal{O}_A(-a))$ is generated by $\det f$ over $q(U)$, we conclude that there is an integer m such that $\mathrm{Nm}(\mathcal{O}_A(-a))$ is isomorphic to $\mathcal{O}_{A/\mu}(-m \cdot \overline{a})$. Pulling back to A shows

$$\begin{aligned} -n &= \deg(\mathcal{O}_A(-a) \otimes \cdots \otimes T^*_{(n-1)\cdot\mu} \mathcal{O}_A(-a)) = \deg(q^* \mathrm{Nm}(\mathcal{O}_A(-a))) = \deg(q) \deg(\mathrm{Nm}(\mathcal{O}_A(-a))) \\ &= -nm, \end{aligned}$$

and thus $m = 1$. □

5.1 Bielliptic surfaces of type 1

Let $\pi \colon A \times B \to S$ be the canonical covering of a bielliptic surface S of type 1. From the classification theorem [1, List 10.27] it follows that there is a 2-torsion element $\tau \in A$ such that S is a quotient of $A \times B$ via the involution

$$\sigma \colon A \times B \to A \times B,\ (a,b) \mapsto (a + \tau, -b).$$

By lemma 5.4, the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Pic}(A \times B)$ equals

$$\sigma^* L(\mathcal{O}_A(n \cdot 0_A) \otimes \alpha, \mathcal{O}_B(m \cdot 0_B) \otimes \beta, \varphi) \cong L(T_\tau^* \mathcal{O}_A(n \cdot 0_A) \otimes \alpha, \mathcal{O}_B(m \cdot 0_B) \otimes \beta^{-1} \otimes (-\varphi)^* P_\tau, -\varphi).$$

Therefore, $L(\mathcal{O}_A(n \cdot 0_A) \otimes \alpha, \mathcal{O}_B(m \cdot 0_B) \otimes \beta, \varphi)$ is anti-invariant if and only if $n = 0 = m$, α is 2-torsion, and $\varphi^* P_\tau \cong \mathcal{O}_B$. Moreover, the map $\mathrm{id} - \sigma^*$ sends $L(\mathcal{O}_A(n \cdot 0_A) \otimes \alpha, \mathcal{O}_B(m \cdot 0_B) \otimes \beta, \varphi)$ to

$$L(P_\tau^n, \beta^2 \otimes (-\varphi)^* P_\tau, 2\varphi).$$

Note that the term $\beta^2 \otimes (-\varphi)^* P_\tau$ can take every value in $\mathrm{Pic}^0(B)$. Therefore, we obtain an isomorphism

$$H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(A \times B)) \cong A[2]/\langle P_\tau \rangle \oplus \mathrm{Hom}_\tau(B, A) \otimes \mathbb{Z}/2\mathbb{Z},$$

where $\mathrm{Hom}_\tau(B, A)$ denotes the subset of those homomorphisms that pull back P_τ to $\mathcal{O}_B$. Since the order of ω_S is prime, we can apply theorem 3.20 to obtain the short exact sequence

$$0 \to \ker(\pi^{Br}) \to H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(A \times B)) \to \ker(\pi^*) \cap \mathrm{im}([2]) \to 0.$$

Observe that the last map sends the unique non-trivial element $\bar{\alpha}$ in $A[2]/\langle P_\tau \rangle$ to the descend of α, by lemma 5.5. Hence, we obtain the isomorphism

$$\ker(\pi^{Br}) \cong \mathrm{Hom}_\tau(B, A) \otimes \mathbb{Z}/2\mathbb{Z}.$$

Theorem 5.6. *[6, Thm. 5.8, 5.10] Let S be a bielliptic surface of type 1 with canonical covering $\pi\colon A \times B \to S$.*

(i) *If A and B are not isogenous, then π^{Br} is injective.*

(ii) *If A and B are isogenous without complex multiplication, then $\ker(\pi^{Br}) \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $\varphi_0^* P_\tau \cong \mathcal{O}_B$, where $\varphi_0\colon B \to A$ generates $\mathrm{Hom}(B, A) \cong \mathbb{Z}$. In this case, the unique non-trivial element in the kernel is represented by the Brauer–Severi variety given as the descent of*

$$\mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi)).$$

Otherwise, $\ker(\pi^{Br}) = 0$.

(iii) *If A and B are isogenous and have complex multiplication, then $\ker(\pi^{Br}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if $\varphi_0^* P_\tau$, $\varphi_1^* P_\tau$ and $(\varphi_0 + \varphi_1)^* P_\tau$ are trivial, where φ_0, φ_1 generate $\mathrm{Hom}(B, A) \cong \mathbb{Z}^2$. In this case, the non-trivial elements in the kernel are represented by the Brauer–Severi varieties given as the descents of*

$$\mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi_0)), \quad \mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi_1)), \quad \text{and} \quad \mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi_0 + \varphi_1)).$$

Furthermore, $\ker(\pi^{Br}) \cong \mathbb{Z}/2\mathbb{Z}$ *if and only if* $\varphi_0^* P_\tau \cong \mathcal{O}_B$, $\varphi_1^* P_\tau \cong \mathcal{O}_B$ *or* $(\varphi_0 + \varphi_1)^* P_\tau \cong \mathcal{O}_B$. *In this case, the unique non-trivial element in the kernel is represented by the Brauer–Severi variety given as the descent of*

$$\mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi_0)), \quad \mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi_1)), \quad \text{or} \quad \mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi_0 + \varphi_1)),$$

respectively. Otherwise, $\ker(\pi^{Br}) = 0$.

Proof. If A and B are not isogenous, then $\mathrm{Hom}(B, A) = 0$. The discussion above then implies that $\ker(\pi^{Br})$ is trivial.

If A and B are isogenous without complex mutliplication, $\mathrm{Hom}(B, A) \cong \mathbb{Z}$, generated by φ_0. In that case, the discussion above implies that $\ker(\pi^{Br})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $\varphi_0^* P_\tau \cong \mathcal{O}_B$, and is trivial otherwise. If $\varphi_0^* P_\tau \cong \mathcal{O}_B$, then the crossed homomorphism

$$\mathbb{Z}/2\mathbb{Z} \to \mathrm{Pic}(A \times B),\ 1 \mapsto L(0, 0, \varphi_0)$$

is mapped to zero by the norm homomorphism. Therefore, the involution constructed in theorem 3.12 induces a descent of $\mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi))$ to a Brauer–Severi variety $P \to S$, which has Brauer class equal to the unique non-trivial element in the kernel of π^{Br}.

If A and B are isogenous and have complex multiplication, the statement is proven by a very similar argument as in the case where A and B do not have complex multiplication. □

We finish by presenting the examples for the respective situations given in [6, Ex. 5.9, 5.11].

(i) If $A = B$ without complex multiplication, then $\varphi_0 = \pm\mathrm{id}$, and thus, $\varphi_0^* P_\tau \cong P_\tau$. Consequently, π^{Br} is injective.

(ii) Let B be without complex multiplication. Choose a 2-torsion point $\theta \in B$ and define $A := B/\theta$. Denote by $q\colon B \to A$ the quotient map and let $\tau \in A$ be the unique point that defines the unique non-trivial line bundle point $P_\tau \in \ker(q^*)$. The triple (A, B, τ) defines a type 1 bielliptic surface with $q^* P_\tau \cong \mathcal{O}_B$. Since q generates $\mathrm{Hom}(B, A)$, the kernel of the Brauer map is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(iii) Define $\Lambda := \langle 1, i\rangle \subset \mathbb{C}$. Multiplication with i on $\mathbb{C}$ preserves this lattice, and therefore we obtain an automorphism $\omega\colon B := \mathbb{C}/\Lambda \to B$ of order 4. Choose a 2-torsion point $\theta \in B$ and define $A := B/\theta$. Denote by $q\colon B \to A$ the quotient map and let $\tau \in A$ be the unique point that defines the unique non-trivial line bundle point $P_\tau \in \ker(q^*)$. The triple (A, B, τ) defines a type 1 bielliptic surface with $q^*P_\tau \cong \mathcal{O}_B$, $\omega^*q^*P_\tau \cong \mathcal{O}_B$, and $(q + q\circ\omega)^*P_\tau \cong \mathcal{O}_B$. Since q and $q\circ\omega$ generate $\mathrm{Hom}(B, A)$, the kernel of the Brauer map is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

(iv) Define $B := \mathbb{C}/\Lambda$ as before, and define $A := B$. For the 2-torsion point τ we choose $\tau := \overline{\frac{1}{2}(1+i)} \in \mathbb{C}/\Lambda$. Note that τ is a fixed point of ω, which implies $(\mathrm{id} + \omega)^*P_\tau \cong \mathcal{O}_B$. Since id and ω generate $\mathrm{Hom}(B, A)$, the kernel of the Brauer map is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

5.2 Bielliptic surfaces of type 3

Let S be a bielliptic surface of type 3. Then there are two elliptic curves A and B, a 4-torsion point $\varepsilon \in A$ and an automorphism $\omega\colon B \to B$ of order 4 such that S is the quotient of $A \times B$ by the $\mathbb{Z}/4\mathbb{Z}$-action

$$\sigma\colon A \times B \to A \times B,\ (a, b) \mapsto (a + \varepsilon, \omega(b)).$$

By applying lemma 5.4, we see that the effect on $\mathrm{Pic}(A \times B)$ is given by

$$\sigma^*L(\mathcal{M}_A, \mathcal{M}_B, \varphi) = L(T_\varepsilon^*\mathcal{M}_A, \omega^*\mathcal{M}_B \otimes \omega^*\varphi^*P_\varepsilon, \varphi\circ\omega).$$

The line bundle $L(\mathcal{O}_A(n\cdot 0_A)\otimes\alpha, \mathcal{O}_B(m\cdot 0_B)\otimes\beta, \varphi)$ is in the kernel of $\mathrm{id} + \sigma^* + (\sigma^2)^* + (\sigma^3)^*$ if and only if the following hold true:

(i) $m = 0 = n$,

(ii) α is a 4-torsion point,

(iii) $\beta\otimes\omega^*\beta\otimes(\omega^2)^*\beta\otimes(\omega^3)^*\beta\otimes\omega^*\varphi^*P_\varepsilon\otimes(\omega^2)^*\varphi^*P_{2\varepsilon}\otimes(\omega^3)^*\varphi^*P_{3\varepsilon} \cong \mathcal{O}_B$, and

(iv) $\varphi + \varphi\circ\omega + \varphi\circ\omega^2 + \varphi\circ\omega^3 = 0$.

Note that ω satisfies $\mathrm{id} + \omega + \omega^2 + \omega^3 = 0$, since $\omega^4 = \mathrm{id}$. Therefore, these conditions simplify to

(i) $m = 0 = n$,

(ii) α is a 4-torsion point, and

(iii) $(\varphi + \varphi\circ\omega)^*P_{2\varepsilon} \cong \mathcal{O}_B$.

On the other hand, the image of $\mathrm{id} - \sigma^*$ consists of line bundles of the form

$$L(P_\varepsilon^n, \beta\otimes\omega^*\beta^\vee\otimes\omega^*\varphi^*P_\varepsilon^\vee, \varphi - \varphi\circ\omega).$$

By choosing β appropriately, the term $\beta\otimes\omega^*\beta^\vee\otimes\omega^*\varphi^*P_\varepsilon^\vee$ can take every value in $\mathrm{Pic}^0(B)$, and hence

$$H^1(\mathbb{Z}/4\mathbb{Z}, \mathrm{Pic}(A\times B)) \cong A[4]/\langle P_\varepsilon\rangle \oplus \mathrm{Hom}_\varepsilon(B, A)/\langle \mathrm{id} - \omega\rangle,$$

where $\mathrm{Hom}_\varepsilon(B, A)$ is defined as the subgroup of those homomorphisms φ that satisfy $(\varphi+\varphi\circ\omega)^*P_{2\varepsilon} \cong \mathcal{O}_B$. Observe, that the norm on $A[4]/\langle P_\varepsilon\rangle$ is injective by lemma 5.5. On the other hand, a morphism $\varphi \in \mathrm{Hom}_\varepsilon(B, A)$ has non-trivial stabilizer if and only if $\varphi = 0$, since $\omega^2 = -\mathrm{id}$. Therefore, by lemma 3.17 the norm is trivial on $\mathrm{Hom}_\varepsilon(B, A)/\langle \mathrm{id} - \omega\rangle$, which shows

$$\ker(\pi^{Br}) \cong \mathrm{Hom}_\varepsilon(B, A)/\langle \mathrm{id} - \omega\rangle.$$

Note that $\mathrm{Hom}(B, B) = \langle \mathrm{id}, \omega\rangle \cong \mathbb{Z}^2$. Thus, if A and B are isogenous, there is an isogeny $\varphi_0\colon B \to A$ such that φ_0 and $\varphi_0\circ\omega$ generate $\mathrm{Hom}(B, A) \cong \mathbb{Z}^2$ by [6, Thm. A.1].

Theorem 5.7. *[6, Thm. 5.14] Let S be a bielliptic surface of type 3 with canonical covering $\pi\colon A \times B \to S$.*

(i) *If A and B are not isogenous, then π^{Br} is injective.*

(ii) *If A and B are isogenous, then π^{Br} is zero if and only if $(\mathrm{id} + \omega)^* \varphi_0^* P_{2\varepsilon}$ is trivial. In this case, the unique non-trivial element in the kernel is represented by the Brauer–Severi variety given as the descent of*

$$\mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi_0)).$$

Proof. The discussion above implies that π^{Br} is zero if and only if there is a non-zero class $\overline{\varphi} \in \mathrm{Hom}_\varepsilon(B, A)/\langle \mathrm{id} - \omega\rangle$. One can write $\varphi = \varphi_1 + \varphi_2$, where φ_1 has even coefficients when written as a linear combination of the generators φ_0, $\varphi_0 \circ \omega$, and φ_2 is either 0, φ_0, $\varphi_0 \circ \omega$, or $\varphi_0 + \varphi_0 \circ \omega$. Since $P_{2\varepsilon}$ is 2-torsion, the condition $(\varphi + \varphi \circ \omega)^* P_{2\varepsilon} \cong \mathcal{O}_B$ reduces to

$$(\varphi_2 + \varphi_2 \circ \omega)^* P_{2\varepsilon} \cong \mathcal{O}_B.$$

Note that φ_1 is contained in $\langle \mathrm{id} - \omega\rangle$. Indeed, $(\varphi_0 + \varphi_0 \circ \omega) \circ (\mathrm{id} - \omega) = 2\varphi_0$, and $(-\varphi_0 + \varphi_0 \circ \omega) \circ (\mathrm{id} - \omega) = 2\varphi_0 \circ \omega$. Thus, since φ_1 has even coefficients, it is contained in $\langle \mathrm{id} - \omega\rangle$. The class of φ in $\mathrm{Hom}_\varepsilon(B, A)/\langle \mathrm{id} - \omega\rangle$ is non-zero, which hence implies that also the class of φ_2 in $\mathrm{Hom}_\varepsilon(B, A)/\langle \mathrm{id} - \omega\rangle$ is non-zero. This excludes the possibilities $\varphi_2 = 0$ and $\varphi_2 = \varphi_0 + \varphi_0 \circ \omega$. Moreover, the fact that $P_{2\varepsilon}$ is 2-torsion implies that $(\varphi_2 + \varphi_2 \circ \omega)^* P_{2\varepsilon} \cong \mathcal{O}_B$ holds for $\varphi_2 = \varphi_0$ if and only if it holds for $\varphi_2 = \varphi_0 \circ \omega$. We conclude that π^{Br} is trivial if and only if $(\varphi_0 + \varphi_0 \circ \omega)^* P_{2\varepsilon} \cong \mathcal{O}_B$.

If $\varphi_0 + \varphi_0 \circ \omega)^* P_{2\varepsilon} \cong \mathcal{O}_B$, then the crossed homomorphism

$$\mathbb{Z}/2\mathbb{Z} \to \mathrm{Pic}(A \times B),\ 1 \mapsto L(0, 0, \varphi_0)$$

is mapped to zero by the norm homomorphism. Therefore, the involution constructed in theorem 3.12 induces a descent of $\mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi))$ to a Brauer–Severi variety $P \to S$, which has Brauer class equal to the unique non-trivial element in the kernel of π^{Br}. □

We finish by presenting the examples for the respective situations given in [6, Ex. 5.15].

(i) Define $\Lambda := \langle 1, i\rangle \subset \mathbb{C}$. Multiplication with i on $\mathbb{C}$ preserves this lattice, and therefore we obtain an automorphism $\omega\colon B := \mathbb{C}/\Lambda \to B$ of order 4. Define $A := B$ and choose $\varepsilon := \overline{\frac{1}{4}(1 + i)} \in \mathbb{C}/\Lambda$. The quadruple $(A, B, \varepsilon, \omega)$ defines a type 3 bielliptic surface. Note that 2ε is a fixed point of ω, and therefore $(\mathrm{id} + \omega)^* P_{2\varepsilon} \cong \mathcal{O}_B$. Since id and ω generate $\mathrm{Hom}(B, A)$, the Brauer map is trivial.

(ii) If we choose $\varepsilon := \overline{\frac{1}{4}}$ instead, the Brauer map is injective, since $(\mathrm{id} + \omega)(2\varepsilon) = \overline{\frac{1}{2}(1 + i)} \neq 0$.

5.3 Bielliptic surfaces of type 5

Let S be a bielliptic surface of type 5. Then there are two elliptic curves A and B, a 3-torsion point $\eta \in A$ and an automorphism $\rho\colon B \to B$ of order 3 such that S is the quotient of $A \times B$ by the $\mathbb{Z}/3\mathbb{Z}$-action

$$\sigma\colon A \times B \to A \times B,\ (a, b) \mapsto (a + \eta, \rho(b)).$$

By applying lemma 5.4, we see that the effect on $\mathrm{Pic}(A \times B)$ is given by

$$\sigma^* L(\mathcal{M}_A, \mathcal{M}_B, \varphi) = L(T_\eta^* \mathcal{M}_A, \rho^* \mathcal{M}_B \otimes \rho^* \varphi^* P_\eta, \varphi \circ \rho).$$

The line bundle $L(\mathcal{O}_A(n \cdot 0_A) \otimes \alpha, \mathcal{O}_B(m \cdot 0_B) \otimes \beta, \varphi)$ is in the kernel of $\mathrm{id} + \sigma^* + (\sigma^2)^*$ if and only if the following hold true:

(i) $m = 0 = n$,

(ii) α is a 3-torsion point,

(iii) $\beta \otimes \rho^*\beta \otimes (\rho^2)^*\beta \otimes \rho^*\varphi^*P_\eta \otimes (\rho^2)^*\varphi^*P_{2\eta} \cong \mathcal{O}_B$, and

(iv) $\varphi + \varphi \circ \rho + \varphi \circ \rho^2 = 0$.

Note that ρ satisfies $\mathrm{id} + \rho + \rho^2 = 0$, since $\rho^3 = \mathrm{id}$. Therefore, these conditions simplify to

(i) $m = 0 = n$,

(ii) α is a 3-torsion point, and

(iii) $(\varphi - \varphi \circ \rho)^*P_\eta \cong \mathcal{O}_B$.

On the other hand, the image of $\mathrm{id} - \sigma^*$ consists of line bundles of the form

$$L(P_\eta^n, \beta \otimes \rho^*\beta^\vee \otimes \rho^*\varphi^*P_\eta^\vee, \varphi - \varphi \circ \rho).$$

By choosing β apropriately, the term $\beta \otimes \rho^*\beta^\vee \otimes \rho^*\varphi^*P_\eta^\vee$ can take every value in $\mathrm{Pic}^0(B)$, and hence

$$H^1(\mathbb{Z}/3\mathbb{Z}, \mathrm{Pic}(A \times B)) \cong A[3]/\langle P_\eta \rangle \oplus \mathrm{Hom}_\eta(B, A)/\langle \mathrm{id} - \rho \rangle,$$

where $\mathrm{Hom}_\eta(B, A)$ is defined as the subgroup of those homomorphisms φ that satisfy $(\varphi - \varphi \circ \rho)^*P_\eta \cong \mathcal{O}_B$. Since the order of ω_S is prime, we can apply theorem 3.20 to obtain the short exact sequence

$$0 \to \ker(\pi^{Br}) \to H^1(\mathbb{Z}/3\mathbb{Z}, \mathrm{Pic}(A \times B)) \to \ker(\pi^*) \cap \mathrm{im}([3]) \to 0.$$

Observe that the last map is an isomorphism on $A[3]/\langle P_\eta \rangle$ by lemma 5.5, and is zero on $\mathrm{Hom}_\eta(B, A)/\langle \mathrm{id} - \rho \rangle$ by lemma 3.17. Thus, we obtain an isomorphism

$$\ker(\pi^{Br}) \cong \mathrm{Hom}_\eta(B, A)/\langle \mathrm{id} - \rho \rangle.$$

Note that $\mathrm{Hom}(B, B) = \langle \mathrm{id}, \rho \rangle \cong \mathbb{Z}^2$. Thus, if A and B are isogenous, there is an isogeny $\varphi_0 \colon B \to A$ such that φ_0 and $\varphi_0 \circ \rho$ generate $\mathrm{Hom}(B, A) \cong \mathbb{Z}^2$ by [6, Thm. A.1].

Theorem 5.8. *[6, Thm. 5.19] Let S be a bielliptic surface of type 5 with canonical covering $\pi \colon A \times B \to S$.*

(i) *If A and B are not isogenous, then π^{Br} is injective.*

(ii) *If A and B are isogenous, then π^{Br} is zero if and only if $(\mathrm{id} - \rho)^*\varphi_0^*P_\eta \cong \mathcal{O}_B$. In this case, the non-trivial elements in the kernel are represented by the Brauer–Severi varieties given as the descents of*

$$\mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi_0) \oplus L(0, 0, \varphi_0 + \varphi_0 \circ \rho)) \quad \text{and} \quad \mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, 2\varphi_0) \oplus L(0, 0, 2\varphi_0 + 2\varphi_0 \circ \rho)).$$

Proof. The discussion above implies that π^{Br} is zero if and only if there is a non-zero class $\overline{\varphi} \in \mathrm{Hom}_\eta(B, A)/\langle \mathrm{id} - \rho \rangle$. One can write $\varphi = \varphi_1 + \varphi_2$, where φ_1 has coefficients divisible by 3 when written as a linear combination of the generators φ_0, $\varphi_0 \circ \omega$, and

$$\varphi_2 \in \{0, \varphi_0, 2\varphi_0, \varphi_0 + \varphi_0 \circ \rho, 2\varphi_0 + \varphi_0 \circ \rho, \varphi_0 + 2\varphi_0 \circ \rho, 2\varphi_0 + 2\varphi_0 \circ \rho, \varphi_0 \circ \rho, 2\varphi_0 \circ \rho\}.$$

Since $P_{2\varepsilon}$ is 3-torsion, the condition $(\varphi - \varphi \circ \rho)^*P_\eta \cong \mathcal{O}_B$ reduces to

$$(\varphi_2 - \varphi_2 \circ \rho)^*P_\eta \cong \mathcal{O}_B.$$

Note that φ_1 is contained in $\langle \mathrm{id} - \omega\rho\rangle$. Indeed, $(2\varphi_0 + \varphi_0 \circ \rho) \circ (\mathrm{id} - \rho) = 3\varphi_0$, and $(-\varphi_0 + \varphi_0 \circ \rho) \circ (\mathrm{id} - \rho) = 3\varphi_0 \circ \rho$. Thus, since φ_1 has coefficients divisible by 3, it is contained in $\langle \mathrm{id} - \rho\rangle$. The class of φ in $\mathrm{Hom}_\eta(B, A)/\langle \mathrm{id} - \rho\rangle$ is non-zero, which hence implies that also the class of φ_2 in $\mathrm{Hom}_\eta(B, A)/\langle \mathrm{id} - \rho\rangle$ is non-zero. This excludes the possibilities $\varphi_2 = 0$, $\varphi_2 = 2\varphi_0 + \varphi_0 \circ \rho$ and $\varphi_2 = \varphi_0 + 2\varphi_0 \circ \rho$. Moreover, the fact that P_η is 3-torsion implies that the condition $(\varphi_2 - \varphi_2 \circ \omega)^* P_{2\varepsilon} \cong \mathcal{O}_B$ holds for $\varphi_2 = \varphi_0$ if and only if it holds for the other six possibilities. We conclude: π^{Br} is trivial if and only if $(\varphi_0 - \varphi_0 \circ \rho)^* P_\eta \cong \mathcal{O}_B$.

If $(\varphi_0 - \varphi_0 \circ \rho)^* P_\eta \cong \mathcal{O}_B$, then the crossed homomorphisms

$$\mathbb{Z}/3\mathbb{Z} \to \mathrm{Pic}(A \times B),\ 1 \mapsto L(0, 0, \varphi_0),\ 2 \mapsto L(0, 0, \varphi_0 \circ \rho)$$
$$\mathbb{Z}/3\mathbb{Z} \to \mathrm{Pic}(A \times B),\ 1 \mapsto L(0, 0, 2\varphi_0),\ 2 \mapsto L(0, 0, 2\varphi_0 \circ \rho)$$

are mapped to zero by the norm homomorphism. Therefore, the involution constructed in theorem 3.12 induces descents of $\mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, \varphi_0) \oplus L(0, 0, \varphi_0 + \varphi_0 \circ \rho))$ and $\mathbb{P}(\mathcal{O}_{A\times B} \oplus L(0, 0, 2\varphi_0) \oplus L(0, 0, 2\varphi_0 + 2\varphi_0 \circ \rho))$ to Brauer–Severi varieties $P_1 \to S$ and $P_2 \to S$, which have Brauer class equal to the two non-trivial element in the kernel of π^{Br}. □

We finish by presenting the examples for the respective situations given in [6, Ex. 5.20].

(i) Define $\Lambda := \langle 1, \zeta := e^{\frac{2\pi i}{3}}\rangle \subset \mathbb{C}$. Multiplication with ζ on $\mathbb{C}$ preserves this lattice, and therefore we obtain an automorphism $\rho\colon B := \mathbb{C}/\Lambda \to B$ of order 3. Define $A := B$ and choose $\eta := \overline{\frac{1}{3} + \frac{2}{3}\zeta} \in \mathbb{C}/\Lambda$. The quadruple (A, B, η, ρ) defines a type 5 bielliptic surface. Note that η is a fixed point of ρ, and therefore $(\mathrm{id} - \rho)^* P_\eta \cong \mathcal{O}_B$. Since id and ρ generate $\mathrm{Hom}(B, A)$, the Brauer map is trivial.

(ii) If we choose $\varepsilon := \overline{\frac{1}{3}}$ instead, the Brauer map is injective, since $(\mathrm{id} - \rho)(\eta) = \overline{\frac{1}{3} - \frac{1}{3}\zeta} \neq 0$.

References

[1] Lucian Bădescu. *Algebraic Surfaces*. Universitext. Springer, 2001.

[2] Wolf P. Barth, Klaus Hulek, Chis A. M. Peters, and Antonius van de Ven. *Compact Complex Surfaces*. Second endlarged edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 2004.

[3] Arnaud Beauville. On the Brauer group of Enriques surfaces. In: *Mathematical Research Letters* 16.6 (2009).

[4] Kenneth S. Brown. *Cohomology of Groups*. Graduate Texts in Mathematics 87. Springer New York, 1982.

[5] David Eisenbud and Joe Harris. *3264 and all that: Intersection theory in algebraic geometry*. Cambridge University Press, 2016.

[6] Eugenia Ferrari, Sofia Tirabassi, Magnus Vodrup, and Jonas Bergström. On the Brauer group of bielliptic surfaces (with an appendix by Jonas Bergström and Sofia Tirabassi). In: *Documenta Mathematica* 27 (2022), pp. 383–425. URL: `https://doi.org/10.4171%2Fdm%2F873`.

[7] Ofer Gabber. *Some theorems on azumaya algebras*. Ed. by Michel Kervaire and Manuel Ojanguren. Springer Berlin Heidelberg, 1981, pp. 129–209. ISBN: 978-3-540-38531-8.

[8] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes*. Springer Studium Mathematik - Master. Springer Spektrum Wiesbaden, 2020.

[9] Alexander Grothendieck. Le groupe de Brauer : I. Algèbres d'Azumaya et interprétations diverses. In: *Séminaire N. Bourbaki* 290 (1966).

[10] Alexander Grothendieck. Le groupe de Brauer : II. Théories cohomologiques. In: *Séminaire N. Bourbaki* 297 (1966).

[11] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002. URL: `https://pi.math.cornell.edu/~hatcher/AT/ATpage.html`.

[12] Raymond T. Hoobler. Brauer groups of abelian schemes. In: *Annales Scientifiques De L Ecole Normale Superieure* 5 (1972), pp. 45–70.

[13] Daniel Huybrechts. *Lectures on K3 surfaces*. Cambridge University Press, 2015.

[14] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves*. 2nd ed. Cambridge Mathematical Library. Cambridge University Press, 2010.

[15] Hajime Kaji. On the Vector Bundles Whose Endomorphisms Azumaya Algebras of Cyclic Type. In: *Journal of Algebra* 117.2 (1988), pp. 297–326. ISSN: 0021-8693. URL: `https://www.sciencedirect.com/science/article/pii/0021869388901093`.

[16] Hoil Kim. Moduli spaces of stable vector bundles on Enriques surfaces. In: *Nagoya Mathematical Journal* 150 (1998), pp. 85–94.

[17] János Kollár. Severi-Brauer varieties; a geometric treatment. In: (2016). arXiv: `1606.04368 [math.AG]`.

[18] Hermes Martínez. The Brauer Group of K3 Covers. In: (2012). URL: `https://api.semanticscholar.org/CorpusID:118129521`.

[19] James S. Milne. *Étale Cohomology (PMS-33), Volume 33*. Princeton: Princeton University Press, 1980. ISBN: 9781400883981. URL: `https://doi.org/10.1515/9781400883981`.

[20] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper's series in modern mathematics. New York, Harper and Row, 1968.

[21] David Mumford. *Abelian varieties*. Vol. 5. Tata Inst. Fundam. Res., Stud. Math. London: Oxford University Press, 1970.

[22] Daniel Naie. Special rank two vector bundles over Enriques surfaces. In: *Mathematische Annalen* 300.2 (1994), pp. 297–316.

[23] Howard Nuer. A note on the existence of stable vector bundles on Enriques surfaces. In: *Selecta Mathematica* 22.3 (Dec. 2015), pp. 1117–1156. URL: `https://doi.org/10.1007%2Fs00029-015-0218-6`.

[24] A. N. Skorobogatov. On the Elementary Obstruction to the Existence of Rational Points. In: *Mathematical Notes* 81 (Feb. 2007). URL: `https://doi.org/10.1134/S0001434607010099`.